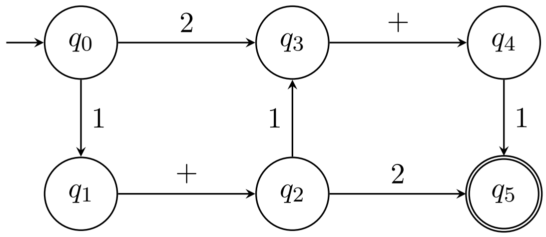
Suggested Solutions

[Compiled on September 4, 2017]

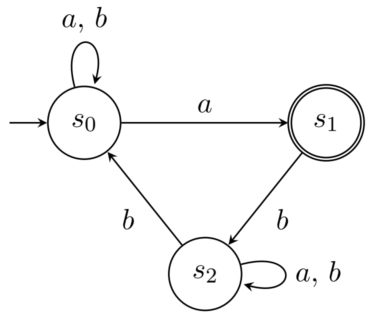
1. Given an alphabet $\{1, 2, +\}$, draw a finite state automaton such that the automaton accepts words evaluated to 3.

   Solution.

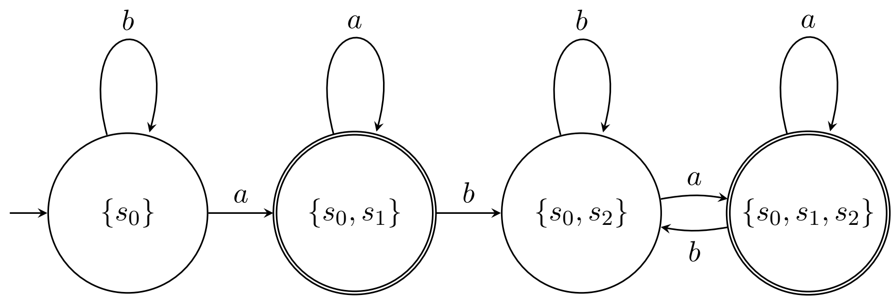


□

2. Apply subset construction to determinize the following automaton.



   Solution.



□

3. Let $M_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$ be two NFAs. Construct an NFA M_3 such that $L(M_3) = L(M_1) \backslash L(M_2)$. Please describe the components of M_3 in detail.

   Solution. Observe that $L(M_3) = L(M_1) \cap (\Sigma^* \backslash L(M_2))$. The automaton M_3 can be obtained by taking the intersection of M_1 and the complement of M_2. The complement of M_2 can be obtained by subset construction followed by complementing accepting states. Define $M_3 = (Q_1 \times 2^{Q_2}, \Sigma, \Delta, I_1 \times \{I_2\}, G)$ where

- $(q', rs') \in \Delta((q, rs), a)$ for all $a \in \Sigma$ if and only if
  - $q' \in \delta_1(q, a)$, and
  - $rs' = \bigcup_{r \in rs} \delta_2(r, a)$, and
- $G = \{(q, rs) \mid q \in F_1, rs \subseteq Q_2, \text{ and } rs \cap F_2 = \varnothing\}$

□

4. Write regular expressions to describe the following languages. ($\Sigma = \{a, b\}$)

   (a) $\{w \mid \text{the length of } w \text{ is even}\}$

   (b) $\{w \mid w \text{ has at most two } b\text{'s}\}$

   (c) $\{w \mid \text{every } a \text{ in } w \text{ is followed by } b\}$

   Solution.

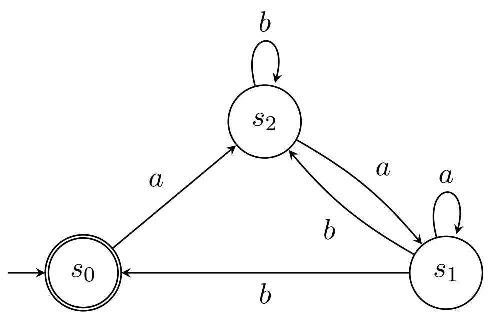
   (a) $(\Sigma\Sigma)^*$

   (b) $(a^*) + (a^*ba^*) + (a^*ba^*ba^*)$

   (c) $(b^*(ab)^*)^*$

□

5. Express the language of the following automaton by a regular expression.



Solution. Define the following equation system.

$$
\begin{aligned}
Q_0 &= aQ_2 + \epsilon & (1)\\
Q_1 &= aQ_1 + bQ_0 + bQ_2 & (2)\\
Q_2 &= aQ_1 + bQ_2 & (3)
\end{aligned}
$$

By equations 2 and 1, we have

$$
\begin{aligned}
Q_1 &= aQ_1 + bQ_0 + bQ_2\\
&= aQ_1 + b(aQ_2 + \epsilon) + bQ_2\\
&= aQ_1 + bQ_2 + b.
\end{aligned}
$$

By Ardens Lemma,

$$Q_1 = a^*(bQ_2 + b). \qquad (4)$$

By equations 3 and 4, we have

$$\begin{aligned} Q_2 &= aQ_1 + bQ_2 \\ &= a(a^*(bQ_2 + b)) + bQ_2 \\ &= (aa^*b + b)Q_2 + aa^*b. \end{aligned}$$

By Ardens Lemma,

$$Q_2 = (aa^*b + b)^*(aa^*b). \tag{5}$$

Finally by equations 1 and 5, we have

$$\begin{aligned} Q_0 &= aQ_2 + \epsilon \\ &= a(aa^*b + b)^*(aa^*b) + \epsilon. \end{aligned}$$

Thus, the language of the automaton can be expressed in the regular expression $a(aa^*b + b)^*(aa^*b) + \epsilon$. □

6. Write WS1S formulas to describe the following words.
   (a) Only a's can occur between any two occurrences of b's
   (b) Has an odd length (please start with $\exists$)

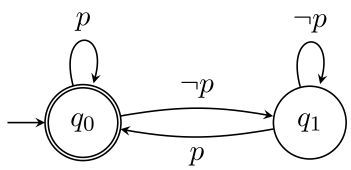
Solution.

(a) $\forall x.\forall y.((P_b(x) \wedge P_b(y) \wedge x < y) \rightarrow (\forall z.(x < z \wedge z < y) \rightarrow P_a(z)))$
(b) $\exists f.\exists l.\exists X.(\mathit{first}(f) \wedge \mathit{last}(l) \wedge X(f) \wedge X(l) \wedge \forall y.\forall z.(S(y,z) \rightarrow (X(y) \leftrightarrow \neg X(z))))$

□

7. Draw a Büchi automaton that accepts infinite words where p holds infinitely many times. ($\Sigma = \{p, \neg p\}$)

Solution.



□

8. Express the following sentences in LTL formulas.
   (a) "p occurs infinitely often"
   (b) "whenever a message is sent, eventually an acknowledgement will be received"

Solution.

(a) $\mathbf{G}\ \mathbf{F}\ p$
(b) $\mathbf{G}(\mathit{sent} \rightarrow \mathbf{F}\ \mathit{ack})$

□