Exercise 2

Optimization Problem

Optimization problem is the problem of finding the best solution from all feasible solutions. One possible instance is the following. Given a constraint F and a target formula f, we want to find a solution s of F such that $f(s) \leq f(s')$, where $s' \neq s$ is any other solution of F.

Questions

Now we assume that $F = (2x + y < 6) \vee (3x < 7 \wedge 2y < 1)$ and $f = -2x - y$.

1. Write a quantifier-free FOL in $T_{\mathbb{Q}}$ over *only* x and y such that the their solutions are also the solutions of optimization problem when the variable domains are real numbers.
2. Do the above in $\widehat{T_{\mathbb{Z}}}$ and over integer domains.

Hint: you can begin with a formula with alternation of quantifiers and do quantifier elimination.

Solution:

First, we construct the formula denoting the optimal solutions that satisfies $F(x, y) = (2x + y < 6) \vee (3x < 7 \wedge 2y < 1)$ and has minimum value in $f(x, y) = -2x - y$. Here, we interpret $A \leq B$ as the shorthand of $\neg(B < A)$

$$\begin{aligned}
Min \equiv\ & F(x,y) \wedge \forall x'.\forall y'.F(x',y') \rightarrow f(x,y) \leq f(x',y') \\
\equiv\ & F(x,y) \wedge \forall x'.\forall y'.F(x',y') \rightarrow \neg(f(x',y') < f(x,y)) \\
\equiv\ & F(x,y) \wedge \neg\exists x'.\exists y'.\neg(F(x',y') \rightarrow \neg(f(x',y') < f(x,y))) \\
\equiv\ & F(x,y) \wedge \neg\exists x'.\exists y'.\neg(\neg F(x',y') \vee \neg(f(x',y') < f(x,y))) \\
\equiv\ & F(x,y) \wedge \neg\exists x'.\exists y'.F(x',y') \wedge f(x',y') < f(x,y)
\end{aligned}$$

Then, we do quantifier elimination on given theory and domain.

2. Do the above in $\widehat{T_{\mathbb{Z}}}$ and over integer domains.

Here, we first depict how to eliminate y' of the inner quantifier.

$$\begin{aligned}
& \exists y'.F(x',y') \wedge f(x',y') < f(x,y) \\
\equiv\ & \exists y'.((2x' + y' < 6) \vee (3x' < 7 \wedge 2y' < 1)) \wedge (-2x' - y' < -2x - y) \\
\equiv\ & \exists y'.((y' < 6 - 2x') \vee (3x' < 7 \wedge 2y' < 1)) \wedge (-2x' + 2x + y < y') \\
\equiv\ & \exists y'.((2y' < 12 - 4x') \vee (3x' < 7 \wedge 2y' < 1)) \wedge (-4x' + 4x + 2y < 2y') \quad \text{Let } A = 2y' \\
\equiv\ & \exists A.((A < 12 - 4x') \vee (3x' < 7 \wedge A < 1)) \wedge (-4x' + 4x + 2y < A) \wedge (2|A)
\end{aligned}$$

Consider *left infinite projection* $F_{-\infty}[A]$

$$\begin{aligned} & \exists A.((A < 12 - 4x') \vee (3x' < 7 \wedge A < 1)) \wedge (-4x' + 4x + 2y < A) \wedge (2|A) \\ \equiv\ & \bigvee_{j=1}^{2}((\top) \vee (3x' < 7 \wedge \top)) \wedge (\bot) \wedge (2|j) \\ \equiv\ & \bot \end{aligned}$$

Consider case with a least number

$$\begin{aligned} & \exists A.((A < 12 - 4x') \vee (3x' < 7 \wedge A < 1)) \wedge (-4x' + 4x + 2y < A) \wedge (2|A) \\ \equiv\ & \bigvee_{j=1}^{2}((-4x' + 4x + 2y + j < 12 - 4x') \vee (3x' < 7 \wedge -4x' + 4x + 2y + j < 1)) \\ & \wedge (-4x' + 4x + 2y < -4x' + 4x + 2y + j) \wedge (2| - 4x' + 4x + 2y + j) \\ \equiv\ & \bigvee_{j=1}^{2}((4x + 2y + j < 12) \vee (3x' < 7 \wedge -4x' + 4x + 2y + j < 1)) \\ & \wedge \top \wedge (2| - 4x' + 4x + 2y + j) \\ \equiv\ & ((4x + 2y < 11) \vee (3x' < 7 \wedge -4x' + 4x + 2y < 0)) \wedge (2| - 4x' + 4x + 2y + 1) \\ & \vee ((4x + 2y < 10) \vee (3x' < 7 \wedge -4x' + 4x + 2y < -1)) \wedge (2| - 4x' + 4x + 2y + 2) \\ \equiv\ & ((4x + 2y < 11) \vee (3x' < 7 \wedge -4x' + 4x + 2y < 0)) \wedge \bot \\ & \vee ((2x + y < 5) \vee (3x' < 7 \wedge -4x' + 4x + 2y < -1)) \wedge \top \\ \equiv\ & (2x + y < 5) \vee (3x' < 7 \wedge -4x' + 4x + 2y < -1) \end{aligned}$$

After y' is eliminated, we further compute how to eliminate x'. First, we know x' is not a free variable in $2x + y < 5$

$$\begin{aligned} & \exists x'.(2x + y < 5) \vee (3x' < 7 \wedge -4x' + 4x + 2y < -1) \\ \equiv\ & (2x + y < 5) \vee \exists x'.(3x' < 7 \wedge 4x + 2y + 1 < 4x') \end{aligned}$$

Therefore we only consider quantification on following formula

$$\begin{aligned} & \exists x'.(3x' < 7 \wedge 4x + 2y + 1 < 4x') \\ \equiv\ & \exists x'.(12x' < 28 \wedge 12x + 6y + 3 < 12x') \quad \text{Let } B = 12x' \\ \equiv\ & \exists B.(B < 28 \wedge 12x + 6y + 3 < B \wedge 12|B) \end{aligned}$$

Consider *left infinite projection* $F_{-\infty}[B]$

$$\begin{aligned} & \exists B.(B < 28 \wedge 12x + 6y + 3 < B \wedge 12|B) \\ \equiv\ & \bigvee_{j=1}^{12}(\top \wedge \bot \wedge 12|j) \\ \equiv\ & \bot \end{aligned}$$

Consider case with a least number

$$\begin{aligned} & \exists B.B < 28 \wedge 12x + 6y + 3 < B \wedge (12|B) \\ \equiv\ & \bigvee_{j=1}^{12}(12x + 6y + 3 + j < 28 \wedge 12x + 6y + 3 < 12x + 6y + 3 + j \\ & \wedge\ 12|12x + 6y + 3 + j) \\ \equiv\ & \bigvee_{j=1}^{12}(12x + 6y + j < 25 \wedge \top \wedge\ 12|6y + 3 + j) \\ \equiv\ & \bot \vee \bigvee_{j=3,9}(12x + 6y + j < 25 \wedge\ 12|6y + 3 + j) \\ \equiv\ & (12x + 6y + 3 < 25 \wedge 12|6y + 3 + 3) \vee (12x + 6y + 9 < 25 \wedge 12|6y + 3 + 9) \\ \equiv\ & (12x + 6y < 22 \wedge 2|y + 1) \vee (12x + 6y < 16 \wedge 2|y) \end{aligned}$$

Hence, the final quantifier-free formula should be

$$\begin{aligned} Min \equiv\ & ((2x + y < 6) \vee (3x < 7 \wedge 2y < 1)) \\ & \wedge \neg(2x + y < 5 \vee (12x + 6y < 22 \wedge 2|y + 1) \vee (12x + 6y < 16 \wedge 2|y)) \end{aligned}$$