Logic

Solutions to Homework for Lecture III

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These are possible solutions to the homework for the second lecture. Some questions can be answered in more than one way, so if your answer differs from mine that does not mean you are wrong. In fact, my solution might be wrong, in which case you should contact me as soon as possible.

1 Logic, Semantics, and Deductive Systems

Show that

- $\vdash_{\mathrm{NJ}} \neg\neg\neg\neg P \to \neg\neg P$ for any propositional letter P.

$$
(\to\mathrm{I}^u)\,\dfrac{(\to\mathrm{I}^w)\,\dfrac{(\to\mathrm{E})\,\dfrac{[u\colon \neg\neg\neg\neg P] \qquad (\to\mathrm{I}^v)\,\dfrac{(\to\mathrm{E})\,\dfrac{[v\colon \neg\neg P] \qquad [w\colon \neg P]}{\bot}}{\neg\neg\neg P}}{\bot}}{\neg\neg P}}{\neg\neg\neg\neg P \to \neg\neg P}
$$

- $u\colon \neg\neg\varphi \to \varphi, v\colon \neg\neg\psi \to \psi \vdash_{\mathrm{NJ}} \neg\neg(\varphi \wedge \psi) \to \varphi \wedge \psi$

$$
(\to\mathrm{I}^w)\,\dfrac{(\wedge\mathrm{I})\,\dfrac{(\to\mathrm{E})\,\dfrac{[u\colon \neg\neg P \to P] \qquad (\to\mathrm{I}^u)\,\dfrac{(\to\mathrm{E})\,\dfrac{[w\colon \neg\neg(P \wedge Q)] \qquad (\to\mathrm{I}^{u'})\,\dfrac{(\to\mathrm{E})\,\dfrac{[u\colon \neg P] \qquad (\wedge\mathrm{E}_l)\,\dfrac{[u'\colon P \wedge Q]}{P}}{\bot}}{\neg(P \wedge Q)}}{\bot}}{\neg\neg P}}{P} \qquad (\to\mathrm{E})\,\dfrac{[v\colon \neg\neg Q \to Q] \qquad (\to\mathrm{I}^v)\,\dfrac{(\to\mathrm{E})\,\dfrac{[w\colon \neg\neg(P \wedge Q)] \qquad (\to\mathrm{I}^{v'})\,\dfrac{(\to\mathrm{E})\,\dfrac{[v\colon \neg Q] \qquad (\wedge\mathrm{E}_r)\,\dfrac{[v'\colon P \wedge Q]}{Q}}{\bot}}{\neg(P \wedge Q)}}{\bot}}{\neg\neg Q}}{Q}}{P \wedge Q}}{\neg\neg(P \wedge Q) \to P \wedge Q}
$$

- Compute $(((P \to Q) \to P) \to P)^*$.

  $(((P \to Q) \to P) \to P)^* \equiv ((\neg\neg P \to \neg\neg Q) \to \neg\neg P) \to \neg\neg P$

  Bonus: Show that $\vdash_{\mathrm{NJ}} (((P \to Q) \to P) \to P)^*$

$$
(\to\mathrm{I}^u)\,\dfrac{(\to\mathrm{I}^v)\,\dfrac{(\to\mathrm{E})\,\dfrac{(\to\mathrm{E})\,\dfrac{[u\colon (\neg\neg P \to \neg\neg Q) \to \neg\neg P] \qquad (\to\mathrm{I}^w)\,\dfrac{(\bot\mathrm{E})\,\dfrac{(\to\mathrm{E})\,\dfrac{[w\colon \neg\neg P] \qquad [v\colon \neg P]}{\bot}}{\neg\neg Q}}{\neg\neg P \to \neg\neg Q}}{\neg\neg P} \qquad [v\colon \neg P]}{\bot}}{\neg\neg P}}{((\neg\neg P \to \neg\neg Q) \to \neg\neg P) \to \neg\neg P}
$$

2 Proof Normalization (Bonus)

Recall the Church encoding of natural numbers as lambda terms: The number n is encoded as the term $\lambda s\colon A \to A.\lambda z\colon A.s^n\,z$, where $s^n\,z$ represents the n-fold application of s to z (for details, see Dr. Chen's lecture notes).

The successor function S for Church numerals can be implemented by the lambda term

$$
\lambda x\colon (A \to A) \to A \to A.\lambda s\colon A \to A.\lambda z\colon A.x\,s\,(s\,z)
$$

Give a derivation corresponding to the term $S\,0$ and normalize it. What proof term does the resulting derivation correspond to?

Derivation corresponding to $S\,0$:

$$
(\to\mathrm{E})\,\dfrac{(\to\mathrm{I}^x)\,\dfrac{(\to\mathrm{I}^s)\,\dfrac{(\to\mathrm{I}^z)\,\dfrac{(\to\mathrm{E})\,\dfrac{(\to\mathrm{E})\,\dfrac{[x\colon (A \to A) \to A \to A] \qquad [s\colon A \to A]}{A \to A} \qquad (\to\mathrm{E})\,\dfrac{[s\colon A \to A] \qquad [z\colon A]}{A}}{A}}{A \to A}}{(A \to A) \to A \to A}}{((A \to A) \to A \to A) \to (A \to A) \to A \to A} \qquad \begin{array}{c}[s'\colon A \to A] \\ (\to\mathrm{I}^{s'})\,\dfrac{(\to\mathrm{I}^{z'})\,\dfrac{[z'\colon A]}{A \to A}}{(A \to A) \to A \to A}\end{array}}{(A \to A) \to A \to A}
$$

Note that there is a detour: the left premise of the final $(\to\mathrm{E})$ is derived using $(\to\mathrm{I}^x)$. We take out the right half of the prooftree and plug it into assumption x.

$$
(\to\mathrm{I}^s)\,\dfrac{(\to\mathrm{I}^z)\,\dfrac{(\to\mathrm{E})\,\dfrac{(\to\mathrm{E})\,\dfrac{\begin{array}{c}[s'\colon A \to A] \\ (\to\mathrm{I}^{s'})\,\dfrac{(\to\mathrm{I}^{z'})\,\dfrac{[z'\colon A]}{A \to A}}{(A \to A) \to A \to A}\end{array} \qquad [s\colon A \to A]}{A \to A} \qquad (\to\mathrm{E})\,\dfrac{[s\colon A \to A] \qquad [z\colon A]}{A}}{A}}{A \to A}}{(A \to A) \to A \to A}
$$

Now we have another detour: the left premise of the topmost, leftmost $(\to\mathrm{E})$ is derived using $(\to\mathrm{I}^{z'})$; so we again remove the detour by plugging:

$$
\begin{prooftree}
\AxiomC{$[s\colon A \to A]$}
\noLine
\UnaryInfC{$[z'\colon A]$}
\LeftLabel{$(\to\mathrm{I}^{z'})$}
\UnaryInfC{$A \to A$}
\AxiomC{$[s\colon A \to A]$}
\AxiomC{$[z\colon A]$}
\LeftLabel{$(\to\mathrm{E})$}
\BinaryInfC{A}
\LeftLabel{$(\to\mathrm{E})$}
\BinaryInfC{A}
\LeftLabel{$(\to\mathrm{I}^{z})$}
\UnaryInfC{$A \to A$}
\LeftLabel{$(\to\mathrm{I}^{s})$}
\UnaryInfC{$(A \to A) \to A \to A$}
\end{prooftree}
$$

There is still one detour left (again in the upper left hand corner), so we need to plug once more:

$$
\begin{prooftree}
\AxiomC{$[s\colon A \to A]$}
\AxiomC{$[z\colon A]$}
\LeftLabel{$(\to\mathrm{E})$}
\BinaryInfC{A}
\LeftLabel{$(\to\mathrm{I}^{z})$}
\UnaryInfC{$A \to A$}
\LeftLabel{$(\to\mathrm{I}^{s})$}
\UnaryInfC{$(A \to A) \to A \to A$}
\end{prooftree}
$$

This corresponds to the lambda term $\lambda s\colon A \to A.\lambda z\colon A.s\,z$, which is the Church encoding of 1.

3 Curry-Howard

> Show that there is no simply typed lambda term M such that $\vdash M\colon (A \to A) \to A$.

Assume there were such a term M. Viewed as a proof term, it would represent a derivation of $(A \to A) \to A$ in NJ, hence we would have $\vdash_{\mathrm{NJ}} (A \to A) \to A$. That would imply $\models (A \to A) \to A$ by the soundness theorem. However, that is not true: for an interpretation I with $I(A) = \mathtt{F}$, we have $[\![(A \to A) \to A]\!]_I = \mathtt{F}$. In conclusion, such an M cannot exist.