First-order logic

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Credits:

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The contents are based on the Slides of Ming-Hsien Tsai, Anthony Lin and David Mantre

Limitations of propositional logic

- Consider the following classical argument:

$$\begin{array}{ll} (1) & \text{All men are mortal} \\ (2) & \text{Socrates is a man} \\ \hline \multicolumn{2}{l}{\text{Therefore: Socrates is mortal}} \end{array}$$

- Can you express this in propositional logic?

Limitations of propositional logic

- Here is an attempt:

Problem:
How big is
this formula?

(1) All men are mortal

$$Man(Socrates) \rightarrow Mortal(Socrates)$$

$$Man(Plato) \rightarrow Mortal(Plato)$$

...

(2) Socrates is a man

$$Man(Socrates)$$

Therefore: Socrates is mortal

$$Mortal(Socrates)$$

A better solution

- Extend the logic to easily refer to "all men"

$$\forall x. (Man(x) \rightarrow Mortal(x))$$

quantifier

- Read (verbose): *"For all x, if x is a man, then x is mortal"*
- Note: Proposition are now "predicates" which depend on x
- Observation: two lines vs. billions of line

What else can you say in FOL?

- There is a man who is not married

$$\exists x. (man(x) \wedge \neg\, married(x))$$

- Every person has a mother

$$\forall x. (person(x) \rightarrow (\exists y. (motherOf(y, x))))$$

- Some person have two mobile phones

$$\exists x \exists y \exists z. (person(x) \wedge mp(y, x) \wedge mp(z, x) \wedge z \neq y)$$

First-order logic (FOL) syntax

Atomic formula

- Examples of "atomic formulas" ("atoms") in FOL:

$$man(x) \qquad even(1) \qquad mp(y, x)$$

variable   Constant   Predicate

- Predicate have arities (# arguments):
  - $man, even$ have arity 1
  - mp has arity 2

Terms (basic element to form atoms)

- Terms: variables/constants/functions-over-terms
  - Variables: $x, y, \ldots$
  - Function symbols (with arities): $f/2, +/2, sin/1, \pi/0, \ldots$
  - Constants (0-ary function): $0, 1, \pi$, "$John$", …

- Predicate symbols (with arities): $man/1, mp/2, =/2$
  - Definition: If R/i is a predicate symbol with arity i and each of $t_1, t_2, \ldots, t_i$ is a term, then $R(t_1, t_2, \ldots, t_i)$ is an atomic formula

Example

- An atomic formula:

$$=(+(pow(x,3), pow(y,n)), pow(z,n))$$

"$=$" is a predicate
"pow" and + are functions
"x", "y", "z", "n" are variables
"3" is a constant

- For $=$ and $+$, we often write in **infix** instead of **prefix** form
  - $pow(x,3) + pow(y,n) = pow(z,n)$

Formulas

- As in boolean logic, build formulas from atomic formulas with Boolean connectives:

$$\neg, \quad \wedge, \quad \vee, \quad \rightarrow, \quad \leftrightarrow$$

- In addition, formulas can be "quantified":

If F is a formula and x is a variable, then

$\forall x. (F)$ Is a formula

$\exists x. (F)$ Is a formula

Exercise

- How do you build the following formulas?

$$\exists x.\, man(x) \land \neg married(x)$$

$$\big(\exists x.\, man(x)\big) \rightarrow \big(\forall y.\, man(x)\big)$$

$$\big(\forall x.\, man(x)\big) \rightarrow \big(\exists y.\, man(x)\big)$$

Exercise

Which are FOL formulas?

- $\exists y \forall x. \big(R(z) \to R(x)\big)$
- $1 + 3 \times 20$
- $pow(x, n) + pow(y, n) = pow(z, n)$
- $\forall x. (\neg\, pow(x, n) \leftrightarrow n = 1)$
- $\exists x \exists f. (f(x) = 0)$

“ $=$ ” , ”R” are predicates
“pow”, “+”, ”f”, “×” are functions

More exercise

- Give a definition of FOL formulas by induction/grammar
- $t ::= f(t_1, \ldots t_n) \mid x \mid c$

x is a variable

Semantics of FOL

Interpretations

- Domains D (a.k.a. universe)
- An assignment function I mapping:
  - Each variable x to an element in D
  - Each function symbol f/n to a n-arity function $\overbrace{D \times \cdots \times D}^{n} \rightarrow D$
  - Each predicate symbol R/n to a n-arity predicate $\overbrace{D \times \cdots \times D}^{n} \rightarrow \mathbb{B}$

Example: Integer Linear Arithmetic (N,+)

- Function symbol: $+/2,\ 0/0, 1/0,\ \ldots$
- Predicate symbol: $=/2$
- Assignment:

$$
\begin{aligned}
&D = \{0,1,2,3,\ldots\} \\
&I(0) = \{(\,) \mapsto 0\},\ \ I(1) = \{(\,) \mapsto 1\},\ldots \\
&I(+) = \{(0,0) \mapsto 0, (0,1) \mapsto 1 \ldots\} \\
&I(=) = \{(0,0) \mapsto T, (0,1) \mapsto F \ldots\}
\end{aligned}
$$

Example: Strings(Σ^*,.)

- Function symbol: $./2,\ \ \mathrm{a}/0, \mathrm{b}/0,\ \ \ldots$
- Predicate symbol: $=/2$
- Assignment:

$$D = \left\{\{a, b, c, d, \ldots\}^*\right\}$$
$$I(a) = \{(\,) \mapsto a\},\ \ I(b) = \{(\,) \mapsto b\}, \ldots$$
$$I(.) = \{(a, a) \mapsto aa, (a, b) \mapsto ab \ldots\}$$
$$I(=) = \{(a, a) \mapsto T, (a, b) \mapsto F \ldots\}$$

Truth depends on interpretations

- The truth/falsehood of an FOL formula depends on interpretations (just as in PL).
- Need to define whether P is true in I ($I \vDash P$, or $I(P) = T$) by induction on P:
  - Atom: $I \vDash R(x, y)$ iff $\big(I(x), I(y)\big) \mapsto T$ is in $I(R)$
  - AND: $I \vDash P \land Q$ iff $I \vDash P$ and $I \vDash Q$
  - OR: $I \vDash P \lor Q$ iff $I \vDash P$ or $I \vDash Q$
  - NOT: $I \vDash \neg P$ iff $I \nvDash F$
- Note: $I\big(f(t_1, \dots, t_n)\big) = I(f)\big(I(t_1), \dots, I(t_n)\big)$

Semantics of ∀ and ∃

Extending $I(P)$ to formulas with quantifiers:

- Forall: $I \vDash \forall x. P \quad \text{iff} \quad I[a/x](P) = T$ for all a in D
- Exists: $I \vDash \exists x. P \quad \text{iff} \quad I[a/x](P) = T$ for some a in D

- Note: $I[a/x] = \{x \mapsto a \ldots\}$

Example 1

$$\varphi \ := \ \underbrace{\forall x \ (x + 5 = x)}_{\text{universal part}} \ \wedge \ \underbrace{\exists y \ (y + 2 \leq y)}_{\text{existential part}}$$

2 Interpretation $I_{\mathbb{Z}}$ (the *standard* integers)

Item	Meaning in $I_{\mathbb{Z}}$
Domain D	all integers $\mathbb{Z}$
"$+$"	usual integer addition
"=", "≤"	usual equality / order
Constant 5	the integer 5

Example 1

$$\varphi \ := \ \underbrace{\forall x \ (x + 5 = x)}_{\text{universal part}} \ \wedge \ \underbrace{\exists y \ (y + 2 \leq y)}_{\text{existential part}}$$

Truth evaluation in $I_{\mathbb{Z}}$

- **Universal part:** $x + 5 = x$ is **false** for every integer, so $I_{\mathbb{Z}} \models \forall x(x + 5 = x)$ is false.
- **Existential part:** $y + 2 \leq y$ is **never true** in $\mathbb{Z}$, so $I_{\mathbb{Z}} \models \exists y(y + 2 \leq y)$ is false.
- **Conjunction rule** both conjuncts are false ⇒ $I_{\mathbb{Z}} \not\models \varphi$.

Example 1

$$\varphi := \underbrace{\forall x\ (x + 5 = x)}_{\text{universal part}} \wedge \underbrace{\exists y\ (y + 2 \leq y)}_{\text{existential part}}$$

3 Interpretation $I_{\mathbb{Z}_5}$ (integers mod 5)

Item	Meaning in $I_{\mathbb{Z}_5}$
Domain D	$\{0, 1, 2, 3, 4\}$
"$+$"	addition *mod 5*
"=", "≤"	ordinary equality / natural order on $\{0, \ldots, 4\}$
Constant 5	the residue 0 (because $5 \equiv 0 \pmod 5$)

Exercise

$$\varphi \;:=\; \underbrace{\forall x \; (x + 5 = x)}_{\text{universal part}} \;\wedge\; \underbrace{\exists y \; (y + 2 \leq y)}_{\text{existential part}}$$

Exercise 2

Consider the following interpretation (social network):

- Predicate: $F/2$
- $D = all\ people\ on\ earth$
- $I(F) = \left\{ \begin{array}{c} (x, y) \mapsto T\!: \ x \text{ is a friend of } y\,, \\ (x, y) \mapsto F\!: \ x \text{ is not a friend of } y \end{array} \right\}$

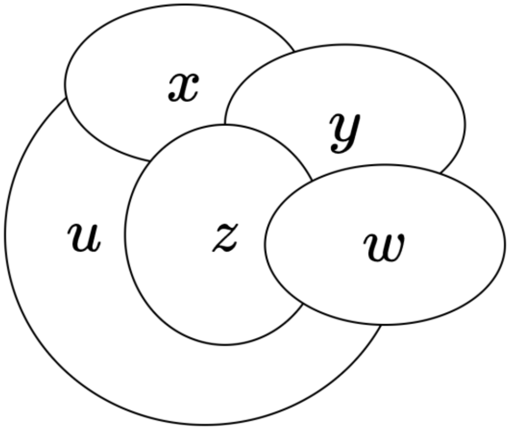
Express (the famous) six-degree of separation:

"The distance between any two people in this graph is six or less"

Exercise 3

Show that it is possible to have 3-coloring for this graph

- Predicate: $\neq/2$
- Variables: u, w, x, y, z
- $D = \{R, G, B\}$



- First: define the interpretation of $\neq$
- Second: formalize the problem in FOL

Exercise 4

In the linear arithmetic $(N, +)$ model, argue (informally) the following formulas are true:

- $\forall x \exists y. (y > x)$
- $\forall x \exists y. y + y = x \vee y + y + 1 = x$

Exercise 5

Consider the interpretation:

$$D = \mathbb{Z}$$

$$I(R) = \{\ (x, y) \mid y = x - z, z = 1,2,3\ \}$$

Is this formula true or false:

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 . \big(R(x_1, y_1) \wedge R(y_1, x_2) \wedge R(x_2, y_2)$$
$$\wedge R(y_2, 0)\big)$$

Try SMT solver

```
(set-logic LIA)

(define-fun R ((x Int) (y Int)) Bool
  (or (= y (- x 1)) (= y (- x 2)) (= y (- x 3)))
)

(assert
  (forall ((x1 Int))
    (exists ((y1 Int))
      (forall ((x2 Int))
        (exists ((y2 Int))
          (and
            (R x1 y1)
            (R y1 x2)
            (R x2 y2)
            (R y2 0)
                        )
        )
      )
    )
  )
)

(check-sat)
```

Exercise 6

Consider the interpretation:

$$D = \mathbb{Z}$$

$$I(R) = \{ (x, y) \mid y = x - z, z = 1,2,3 \}$$

Is this formula true or false:

$$\forall x. \Big((\exists w. 4w = x) \rightarrow \forall z \exists y. \big(R(x, z) \rightarrow R(z, y) \land (\exists w. 4w = y) \big) \Big)$$

Note: $4w$ is a "macro" for $w + w + w + w$ (even this is a macro)

Try SMT solver

```
(set-logic LIA)
(define-fun R ((x Int) (y Int)) Bool
  (or (= y (- x 1)) (= y (- x 2)) (= y (- x 3)))
)

(assert
  (forall ((x Int))
    (exists ((w Int))
      (=>
        (= (+ w w w w) x)
        (forall ((z Int))
          (exists ((y Int))
            (=>
              (R x z)
              (and
                (R z y)
                (exists ((w Int)) (= (+ w w w w) y))
              )
            )
          )
        )
      )
    )
  )
)

(check-sat)
```

Satisfiability / Validity / Equivalence

Satisfiability/validity/ (semantic) equivalence

- A formula is satisfiable if it is true in some interpretation
- A formula is valid if it is true in all interpretations
- Two formulas are equivalent if their truth values are the same under all interpretations

Exercises

Show that all the following examples are satisfiable!

$$\exists x. man(x) \land \neg married(x)$$

$$\left(\exists x. man(x)\right) \rightarrow \left(\forall y. man(x)\right)$$

$$\left(\forall x. man(x)\right) \rightarrow \left(\exists x. man(x)\right)$$

$$\exists x. (man(x) \land \neg man(x))$$

Exercises

Point out valid and invalid formulas!

$$\exists x. man(x) \wedge \neg married(x)$$

$$\left(\exists x. man(x)\right) \rightarrow \left(\forall y. man(x)\right)$$

$$\left(\forall x. man(x)\right) \rightarrow \left(\exists x. man(x)\right)$$

$$\forall x. man(x) \rightarrow man(x)$$

More exercises

- Whether formulas below are valid or not, why?
  - $\forall x.\left(Man(x) \to Mortal(x)\right) \land Man(Socrates) \to Mortal(Socrates)$
  - $(\exists x.P(x) \land \exists x.R(x)) \to \left(\exists x.P(x) \land R(x)\right)$

Some equivalences

- Equivalences from boolean logic carry over to FOL
- New ones, e.g. De Morgan's Laws for FOL:

$$\neg\exists x. \neg F \quad \equiv \quad \forall x. F$$

$$\neg\forall x. \neg F \quad \equiv \quad \exists x. F$$

$\equiv$ means equivalent

Exercise

Prove De Morgan‘s Laws!

(hint: unfold the semantics of the formulas)

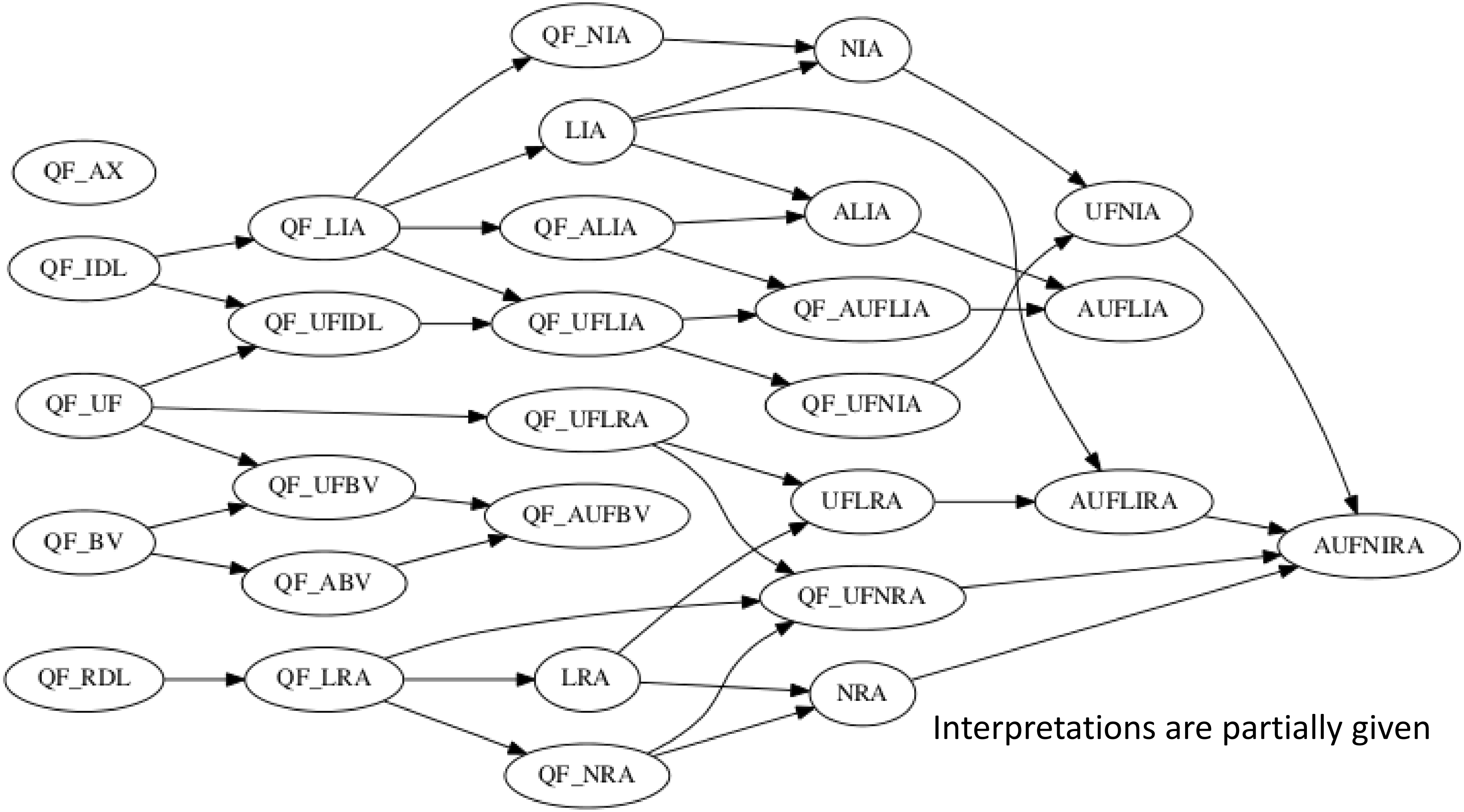
Ponderables

- What's the connection between satisfiability/validity/equivalence?
- Could you give an algorithm for checking satisfiability/validity/equivalence?
- What about the same problem over "finite interpretations"? Over "finite interpretations of size k"?

Ponderables

- What's the connection between satisfiability/validity/equivalence?
- Could you give an algorithm for checking satisfiability/validity/equivalence? No!

Roadmap for FOL after this



Interpretations are partially given

LIA (Linear Integer Arithmetic)

- Function symbol: $+/2,\ \ -/1,\ 0/0, 1/0,\ \ \ldots$
- Predicate symbol: $=/2,\ \ </2$
- Assignment:

Interpretations are partially given

$$D = \mathbb{Z}$$
$$I(0) = \{(\,) \mapsto 0\},\ \ I(1) = \{(\,) \mapsto 1\}, \ldots$$
$$I(+) = \{(n, m) \mapsto n +_{\mathbb{Z}} m\}$$
$$I(-) = \{n \mapsto -_{\mathbb{Z}} n\}$$
$$I(=) = \begin{Bmatrix} (n, n) \mapsto T, \\ \text{otherwise} \mapsto F \end{Bmatrix}$$
$$I(<) = \{(n, m) \mapsto n <_{\mathbb{Z}} m\}$$

An Alternative Way to fix partial interpretation … Axiomization!

$+$ $\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$ (associativity)

$\forall x \forall y (x + y = y + x)$(commutativity)

$\forall x (x + 0 = x)$(identity)

$-$ $\forall x (x + (-x) = 0)$(inverse)

$<$ $\forall x (\neg (x < x))$(irreflexivity)

$\forall x \forall y (x < y \lor x = y \lor y < x)$(trichotomy)

$\forall x \forall y \forall z (x < y \land y < z \rightarrow x < z)$(transitivity)

$\forall x \forall y \forall z (x < y \rightarrow x + z < y + z)$(additive monotonicity)

$0 < 1 \land \forall x (0 < x \rightarrow 1 \leq x)$(minimal positive element)

Roadmap

- Quantifier-free FOL
- Specialized interpretations: linear arithmetic, theory of strings (?), theory of arrays (?), …

Interpretations are partially given

Some more tutorial questions

Free variables

Define this by induction on formula F:

- $free\big(R(x,y)\big) = \{x,y\}$
- $free(F \wedge F') = free(F) \cup free(F')$
- $free(\neg F) = free(F)$
- $free(\forall x. F) = free(F) \setminus \{x\}$
- $free(\exists x. F) = free(F) \setminus \{x\}$

Exercises

What are the free variables of the formulas:

$$\exists x.\, even(x)$$

$$\big(\forall x.\, R(x)\big) \wedge Z(x)$$

More equivalences

If x is not free in the formula G, then:

$$(\forall x. F) \land G \;\equiv\; \forall x. (F \land G)$$

$$(\exists x. F) \land G \;\equiv\; \exists x. (F \land G)$$