# Logic

## I. Intuitionistic propositional logic and natural deduction

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## An elementary theorem and its proof

Logic is about the structures of various kinds of reasoning; in this course we will focus on mathematical reasoning.

**Theorem.** If a function is bijective, it has a two-sided inverse.

**Proof.** Suppose that $f : A \to B$ is bijective. Surjectivity of $f$ implies that there is a function $g : B \to A$ such that $f(g\,b) = b$ for all $b : B$. It remains to show that $g(f\,a) = a$ for all $a : A$, which holds because

$$
\begin{aligned}
& g(f\,a) = a \\
\Leftarrow \quad & \{\, f \text{ injective} \,\} \\
& f(g(f\,a)) = f\,a \\
\Leftarrow \quad & \{\, \text{substituting } f\,a \text{ for } b \,\} \\
& f(g\,b) = b \quad \forall b : B \qquad \qquad \square
\end{aligned}
$$

**Questions.** Why is this a correct proof? What 'game' are we playing? (Is this game meaningful?)

# How to prove an elementary theorem

A mathematical statement should be a proposition in a logical language. For example, the theorem statement actually means

$\forall f : A \to B.$
  $(\forall a : A.\ \forall a' : A.\ f\ a = f\ a' \Rightarrow a = a') \land (\forall b : B.\ \exists a : A.\ f\ a = b)$
    $\Rightarrow \exists g : B \to A.$
      $(\forall b : B.\ f\ (g\ b) = b) \land (\forall a : A.\ g\ (f\ a) = a)$

Then follow the rules of logic to prove the proposition. For example, to prove

- '$\forall x : A.\ P$': Suppose that $x$ is an arbitrary element of $A$. Continue to prove that $P$ holds for $x$.
- '$\exists x : A.\ P$': Find an element of $A$ and prove that $P$ holds for that element.
- '$P \Rightarrow Q$': Suppose that $P$ is true. Continue to prove $Q$.
- '$P \land Q$': Give two proofs respectively proving $P$ and $Q$.
- …

# Formal logic...

... studies the general forms of propositions and valid inferences, ...

- For an extreme example, the truth of the following proposition is determined by the way we use the connectives alone:

  **if** *herba viridi* **and** *area est infectum*, **then** *area est infectum*

  The actual meanings/structures of the two propositions '*herba viridi*' and '*area est infectum*' do not matter.

... and emphasises *formalisation*: writing everything down in symbols following strict rules.

- Concise and unambiguous
- Amenable to mathematical treatment
- Mechanisable: offering an objective foundation (Hilbert's programme); correctness guaranteed by a computer

# Propositional logic

We start with the *propositional connectives* (constants):
conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$), and falsity ($\bot$).

Formally, define a set of *formulas* representing generic propositions
built up from the connectives.

**Definition.** Given a set $\mathcal{PV}$ of propositional variable names, the
set PROP of propositional formulas is *inductively* defined by the
following rules:

- $\mathcal{PV} \subseteq \text{PROP}$;
- $\bot \in \text{PROP}$;
- if $\varphi$, $\psi \in \text{PROP}$, then $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \rightarrow \psi \in \text{PROP}$.

Think of this as a datatype definition in a functional language:

```
type PV   = String
data Prop = Var PV | Falsity
          | Conj Prop Prop | Disj Prop Prop | Impl Prop Prop
```

# Intuitionistic meaning of proposition

A proposition (à la Heyting) expresses an intention/expectation of a *proof*. A proposition is said to be true if an intended/expected proof exists.

To explain the meaning of a proposition, we describe what counts as a proof of that proposition.

- A proof of $\varphi \wedge \psi$ is a proof of $\varphi$ and a proof of $\psi$.
- A proof of $\varphi \vee \psi$ is either a proof of $\varphi$ or a proof of $\psi$.
- A proof of $\varphi \rightarrow \psi$ is a way of constructing a proof of $\psi$ given a proof of $\varphi$.
- There is no proof of $\bot$.

# Formalising propositional deduction: judgement

When constructing a proof, we shift from one state of mind to another, keeping track of what are assumed to be true and what is left to be proved. This 'state of mind' can be formalised as a *judgement*

$$\Gamma \vdash \varphi$$

where $\Gamma$ is a list of propositions assumed to be true, and $\varphi$ is a proposition we wish to prove.

**Example.** $(A \wedge B) \rightarrow (B \wedge A)$ is a proposition, and we write the judgement $\vdash (A \wedge B) \rightarrow (B \wedge A)$ to say that we expect the proposition to be true without assuming anything.

**Question.** Why are we putting parentheses into the formula above?

# Formalising propositional inference: inference rule

A valid shift from one state of mind to another is formalised as an *inference rule*

$$\frac{J_0 \qquad \cdots \qquad J_{n-1}}{J}$$

relating the judgements above the line, called the *premises*, and the judgement below the line, called the *conclusion*, such that when the premises are correct, the conclusion is also correct.

**Example.** There is an inference rule

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} \; (\rightarrow\!\mathsf{I})$$

which can be instantiated to

$$\frac{\mathtt{A} \wedge \mathtt{B} \vdash \mathtt{B} \wedge \mathtt{A}}{\vdash (\mathtt{A} \wedge \mathtt{B}) \rightarrow (\mathtt{B} \wedge \mathtt{A})} \; (\rightarrow\!\mathsf{I})$$

That is, if we wish to deduce the truth of $(\mathtt{A} \wedge \mathtt{B}) \rightarrow (\mathtt{B} \wedge \mathtt{A})$, it suffices to deduce the truth of $\mathtt{B} \wedge \mathtt{A}$ supposing the truth of $\mathtt{A} \wedge \mathtt{B}$.

## Formalising propositional inference: derivation

Starting from a judgement, we apply inference rules until no premises are left. The resulting tree of the applications of inference rules is called a *derivation* of the judgement.

**Example.** Below is a derivation that explains why the proposition $(A \wedge B) \to (B \wedge A)$ is true:

$$\cfrac{\cfrac{\overline{A \wedge B \vdash A \wedge B} \ (\text{assum})}{A \wedge B \vdash B} \ (\wedge ER) \qquad \cfrac{\overline{A \wedge B \vdash A \wedge B} \ (\text{assum})}{A \wedge B \vdash A} \ (\wedge EL)}{\cfrac{A \wedge B \vdash B \wedge A}{\vdash (A \wedge B) \to (B \wedge A)} \ (\to I)} \ (\wedge I)$$

**Question.** How do you translate the above derivation into a natural language?

**Question.** Can you spot propositions, judgements, inference rules, and derivations in pages I-1 and I-2?

**Definition.** A proposition $\varphi$ is called a *theorem* exactly when $\vdash \varphi$ is derivable.

# Natural deduction

Invented by Gentzen, natural deduction is a collection of inference rules intended to capture the natural way in which mathematicians construct proofs. The intuitionistic variant we are introducing is named $\mathrm{NJ}$ by Gentzen.

The simplest inference rule is the assumption rule:

$$\frac{}{\Gamma \vdash \varphi} \text{ (assum)}$$

with the *side condition* that $\varphi$ appears in $\Gamma$.

For each propositional connective (constant), there are

- zero or more *introduction* rules saying how to build a proof involving the connective, and
- zero or more *elimination* rules saying how to use a proof involving the connective.

# Conjunction

- Introduction:
$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \ (\wedge I)$$

- Elimination:
$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \ (\wedge EL) \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi} \ (\wedge ER)$$

**Exercise.** Derive

$$(A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$$

## Implication

- Introduction:

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \ (\to\mathsf{I})$$

- Elimination:

$$\frac{\Gamma \vdash \varphi \to \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \ (\to\mathsf{E})$$

**Exercise.** Derive

$$\vdash (\mathtt{A} \to (\mathtt{B} \wedge \mathtt{C})) \to ((\mathtt{A} \to \mathtt{B}) \wedge (\mathtt{A} \to \mathtt{C}))$$

We will use $\varphi \leftrightarrow \psi$ as a shorthand for $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

**Question.** What is the difference between '$\to$' and '$\vdash$'?

## Disjunction

- Introduction:

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \; (\vee\mathsf{IL}) \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \; (\vee\mathsf{IR})$$

- Elimination:

$$\frac{\Gamma \vdash \varphi \vee \psi \qquad \Gamma, \varphi \vdash \vartheta \qquad \Gamma, \psi \vdash \vartheta}{\Gamma \vdash \vartheta} \; (\vee\mathsf{E})$$

**Exercise.** Derive

$$\vdash (\mathtt{A} \vee \mathtt{B}) \rightarrow (\mathtt{B} \vee \mathtt{A})$$

## Falsity

- Introduction: none
- Elimination:

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \;(\bot \mathsf{E})$$

We use $\neg\varphi$ as a shorthand for $\varphi \to \bot$ and $\top$ for $\bot \to \bot$.

**Exercise.** Derive

$$\vdash ((A \lor B) \land \neg B) \to A$$

**Question.** How do you justify the $\bot$-elimination rule?

# Non-provable propositions

| We can prove | but not |
|---|---|
| $\neg\neg(A \lor \neg A)$ | $A \lor \neg A$ <br> (*law of excluded middle*) |
| $A \to \neg\neg A$ | $\neg\neg A \to A$ <br> (*principle of indirect proof*) |
| $(\neg A \lor \neg B) \to \neg(A \land B)$ | $\neg(A \land B) \to (\neg A \lor \neg B)$ |
| $(A \to B) \to (\neg B \to \neg A)$ | $(\neg B \to \neg A) \to (A \to B)$ |

**Question.** Do you accept the law of excluded middle?

NJ is a formal deduction system for Brouwer's *intuitionism*, where mathematical objects are (subjective) *mental constructions* rather than some kind of Platonic entities that exist independently of the subject's mind.

# From intuitionistic logic to classical logic

We can obtain a natural deduction system $\mathrm{NK}$ for classical logic by adding to $\mathrm{NJ}$ an inference rule encoding the law of excluded middle or the principle of indirect proof (double negation elimination):

$$\frac{}{\Gamma \vdash \varphi \vee \neg\varphi} \ (\mathsf{LEM}) \qquad \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi} \ (\neg\neg\mathsf{E})$$

**Question.** Why does adding one of the two rules suffice?

## Glivenko's theorem

Intuitionistic logic is more precise about constructivity: In intuitionistic logic, $A$ and $\neg\neg A$ have different meanings, but in classical logic they are indistinguishable.

**Theorem.** $\Gamma \vdash_{\mathrm{NK}} \varphi$ if and only if $\neg\neg\Gamma \vdash_{\mathrm{NJ}} \neg\neg\varphi$.

**Proof sketch.** ($\Leftarrow$) Every proposition in $\neg\neg\Gamma$ is implied by the corresponding one in $\Gamma$, so $\Gamma \vdash_{\mathrm{NK}} \neg\neg\varphi$ is derivable, which implies that $\Gamma \vdash_{\mathrm{NK}} \varphi$ is derivable by double negation elimination.

($\Rightarrow$) By induction on $\mathrm{NK}$ derivations. $\square$

**Question.** Shouldn't adding a rule to a system make it stronger (able to derive more propositions) and more useful?