Lambda Calculus and Types: Homework

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1. (20%) Let $f\colon \mathcal{T} \to S$ be a function satisfying $f(\mathtt{0}) = f_{\mathtt{0}}(*)$, $f(\mathtt{succ}\ n) = f_{\mathtt{succ}}(f(n))$, and $f(\mathtt{add}\ n\ m) = f_{\mathtt{add}}(f(n), f(m))$. Show that f is the unique function satisfying

   (a) $f(\mathtt{0}) = f_{\mathtt{0}}(*)$,

   (b) $f(\mathtt{succ}\ n) = f_{\mathtt{succ}}(f(n))$, and

   (c) $f(\mathtt{add}\ n\ m) = f_{\mathtt{add}}(f(n), f(m))$.

   That is, for every $g\colon \mathcal{T} \to S$ satisfying the same set of conditions, $f = g$ must hold.

2. (20%) Show that for every relation R its reflexive and transitive closure R^* is indeed reflexive and transitive. If unsure, use Definition 0.3.

3. (40%) Evaluate the following terms step by step (with $\longrightarrow_{\beta 1}$):

   (a) $\mathtt{add}\ \mathbf{c}_1\ \mathbf{c}_1$

   (b) $\mathtt{pred}\ \mathbf{c}_2$ where $\mathtt{pred} := \lambda n.\, \lambda f x.\, n (\lambda g h.\, h\,(g\,f))\,(\lambda u.\, x)\,(\lambda u.\, u)$

   (c) $\mathtt{sum}\ \mathbf{c}_2$

   (d) $\mathtt{not}\ (\mathtt{and}\ \mathtt{True}\ \mathtt{False})$

4. (20%) Show that if a given λ-term M can reduce to N_1 and N_2 respectively, i.e. $M \longrightarrow_{\beta *} N_1$ and $M \longrightarrow_{\beta *} N_2$, where N_1 and N_2 are in normal form, then $N_1 = N_2$.

Definition 0.1 (Relation composition)**.** Given relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, the *composite* of R and S is defined by

$$R \circ S := \{ (x, z) \in X \times Z \mid \exists y.\ x\ R\ y \quad \text{and} \quad y\ S\ z \}.$$

Definition 0.2 (Diagonal)**.** Given any set X, the *diagonal* relation is a relation $\Delta \subseteq X \times X$ defined by

$$\Delta_X := \{ (x, x) \in X \times X \mid x \in X \}.$$

That is, Δ_X is exactly the equality relation '=': $x_1\ \Delta_X\ x_2 \iff x_1 = x_2$.

Definition 0.3 (Reflexive and transitive closure)**.** Given any relation $R \subseteq X \times X$, the *reflexive and transitive closure* R^* is a relation defined by

$$R^* := \Delta_X \cup \bigcup_{i=1}^{\infty} R^i$$

where $R^1 = R$ and $R^{n+1} = R \circ R^n$.