Suggested Solutions #1

[Compiled on September 5, 2017]

1. Use the semantic method to argue the validity of the following Σ_E-formulae, or identify a counterexample (a falsifying T_E-interpretation).

   (a) $f(x, y) = f(y, x) \rightarrow f(a, y) = f(y, a)$

   (b) $f(g(x)) = g(f(x)) \land f(g(f(y))) = x \land f(y) = x \rightarrow g(f(x)) = x$

   Solution.

   (a) There is a falsifying interpretation where $f(m, n) = m^n$ for all $m, n \in \mathbb{N}$, $x = 2$, $y = 2$, and $a = 3$.

   (b) Assume there is an interpretation M such that $M \not\models f(g(x)) = g(f(x)) \land f(g(f(y))) = x \land f(y) = x \rightarrow g(f(x)) = x$. Then,

$$
\begin{array}{rlcll}
1. & M & \models & f(g(x)) = g(f(x)) \land f(g(f(y))) = x \land f(y) = x & \\
2. & M & \not\models & g(f(x)) = x & \\
3. & M & \models & f(y) = x & \text{(by 1)} \\
4. & M & \models & g(f(y)) = g(x) & \text{(by 3 and function congruence)} \\
5. & M & \models & f(g(f(y))) = f(g(x)) & \text{(by 4 and function congruence)} \\
6. & M & \models & f(g(f(y))) = x & \text{(by 1)} \\
7. & M & \models & f(g(x)) = f(g(f(y))) & \text{(by 5 and symmetry)} \\
8. & M & \models & f(g(x)) = x & \text{(by 6, 7 and transitivity)} \\
9. & M & \models & f(g(x)) = g(f(x)) & \text{(by 1)} \\
10. & M & \models & g(f(x)) = f(g(x)) & \text{(by 9 and symmetry)} \\
11. & M & \models & g(f(x)) = x & \text{(by 10, 8 and transitivity)}
\end{array}
$$

   Since we find a contradiction, the formula is T_E-valid.

   □

2. Given the following 3×3 grid, we would like to find a way to fill the grid with numbers from 1 to 9 such that

   - summations of every row, every column, and every diagonal are the same, and
   - each number can appear only once.

   Try to write an SMT formula such that the way exists if the SMT formula is satisfiable.

Solution. Let $x_{i,j}$ denote the number in the cell at i-th row and j-th column. Assume that there is a sum *sum*. Each number can be from 1 to 9.

$$\bigwedge_{i=\{1,2,3\},j=\{1,2,3\}} (\bigvee_{1\leq k\leq 9} x_{i,j} = k) \tag{Range}$$

Summations of every row, every column, and every diagonal are the same.

$$\begin{array}{ll} & (\bigwedge_{1\leq i\leq 3}(+_{1\leq j\leq 3}x_{i,j} = sum)) \\ \wedge & (\bigwedge_{1\leq j\leq 3}(+_{1\leq i\leq 3}x_{i,j} = sum)) \\ \wedge & (\bigwedge_{i\in\{1,3\}}(x_{1,i} + x_{2,2} + x_{3,4-i} = sum)) \end{array} \tag{Equal}$$

Each number can appear only once.

$$\bigwedge_{1\leq i,j,n,m\leq 3} ((i = n \wedge j = m) \vee (x_{i,j} \neq x_{n,m})) \tag{Distinct}$$

Then, we can find a solution if the SMT formula $Range \wedge Equal \wedge Distinct$ is satisfiable. □

3. Apply the decision procedure for T_E to the following Σ_E-formulae. Provide a level of details as in slides.
   (a) $f(x, y) = f(y, x) \wedge f(a, y) \neq f(y, a)$
   (b) $f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \wedge g(f(x)) \neq x$
   (c) $f(f(f(a))) = f(f(a)) \wedge f(f(f(f(a)))) = a \wedge f(a) \neq a$
   (d) $p(x) \wedge f(f(x)) = x \wedge f(f(f(x))) = x \wedge \neg p(f(x))$

Solution.

(a)

$$\{\{a\}, \{x\}, \{y\}, \{f(x, y)\}, \{f(y, x)\}, \{f(a, y)\}, \{f(y, a)\}\}$$
$$\{\{a\}, \{x\}, \{y\}, \{f(x, y), f(y, x)\}, \{f(a, y)\}, \{f(y, a)\}\} \qquad (f(x, y) = f(y, x))$$

T_E-satisfiable

(b)

$$\{\{x\}, \{y\}, \{f(x)\}, \{g(x)\}, \{f(y)\}, \{f(g(x))\}, \{g(f(x))\}, \{g(f(y))\}, \{f(g(f(y)))\}\}$$
$$(f(g(x)) = g(f(x)))$$
$$\{\{x\}, \{y\}, \{f(x)\}, \{g(x)\}, \{f(y)\}, \{f(g(x)), g(f(x))\}, \{g(f(y))\}, \{f(g(f(y)))\}\}$$
$$(f(g(f(y))) = x)$$
$$\{\{x, f(g(f(y)))\}, \{y\}, \{f(x)\}, \{g(x)\}, \{f(y)\}, \{f(g(x)), g(f(x))\}, \{g(f(y))\}\}$$
$$(f(y) = x)$$
$$\{\{x, f(g(f(y))), f(y)\}, \{y\}, \{f(x)\}, \{g(x)\}, \{f(g(x)), g(f(x))\}, \{g(f(y))\}\}$$
$$(\text{function congruence})$$
$$\{\{x, f(g(f(y))), f(y), f(g(x)), g(f(x))\}, \{y\}, \{f(x)\}, \{g(x), g(f(y))\}\}$$

T_E-unsatisfiable

(c)

$$\begin{array}{ll}
\{\{a\},\{f(a)\},\{f(f(a))\},\{f(f(f(a)))\},\{f(f(f(f(a))))\}\} & \\
\{\{a\},\{f(a)\},\{f(f(a)),f(f(f(a)))\},\{f(f(f(f(a))))\}\} & (f(f(f(a)))=f(f(a))) \\
\{\{a\},\{f(a)\},\{f(f(a)),f(f(f(a))),f(f(f(f(a))))\}\} & \text{(function congruence)} \\
\{\{a,f(f(a)),f(f(f(a))),f(f(f(f(a))))\},\{f(a)\}\} & (f(f(f(f(a))))=a) \\
\{\{a,f(f(a)),f(f(f(a))),f(f(f(f(a)))),f(a)\}\} & \text{(function congruence)}
\end{array}$$

T_E-unsatisfiable

(d) Consider the formula $f_p(x) = \bullet \wedge f(f(x)) = x \wedge f(f(f(x))) = x \wedge f_p(f(x)) \neq \bullet$ instead.

$$\begin{array}{ll}
\{\{\bullet\},\{x\},\{f(x)\},\{f_p(x)\},\{f(f(x))\},\{f_p(f(x))\},\{f(f(f(x)))\}\} & \\
\{\{\bullet,f_p(x)\},\{x\},\{f(x)\},\{f(f(x))\},\{f_p(f(x))\},\{f(f(f(x)))\}\} & (f_p(x)=\bullet) \\
\{\{\bullet,f_p(x)\},\{x,f(f(x))\},\{f(x)\},\{f_p(f(x))\},\{f(f(f(x)))\}\} & (f(f(x))=x) \\
\{\{\bullet,f_p(x)\},\{x,f(f(x))\},\{f(x),f(f(f(x)))\},\{f_p(f(x))\}\} & \text{(function congruence)} \\
\{\{\bullet,f_p(x)\},\{x,f(f(x)),f(x),f(f(f(x)))\},\{f_p(f(x))\}\} & (f(f(f(x)))=x) \\
\{\{\bullet,f_p(x),f_p(f(x))\},\{x,f(f(x)),f(x),f(f(f(x)))\}\} & \text{(function congruence)}
\end{array}$$

T_E-unsatisfiable

□

4. Apply the decision procedure for T_{cons} to the following T_{cons}-formulae. Please write down the call sequence to the `MERGE` procedure, draw the final DAG, and draw the final DAG.

   (a) $car(x) = y \wedge cdr(x) = z \wedge x \neq cons(y, z)$

   (b) $\neg atom(x) \wedge car(x) = y \wedge cdr(x) = z \wedge x \neq cons(y, z)$

Solution.

(a) The following is the initial DAG.

The following is the merge sequences.

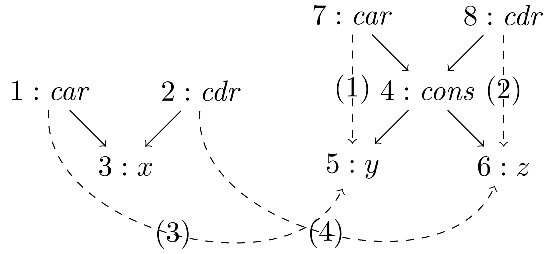
(1) Add node $7 : car(cons(y, z))$ and `MERGE` 7 5 (by left projection)

(2) Add node $8 : cdr(cons(y, z))$ and `MERGE` 8 6 (by right projection)

(3) `MERGE` 1 5 (by $car(x) = y$)

(4) `MERGE` 2 6 (by $cdr(x) = z$)

The following is the final DAG.

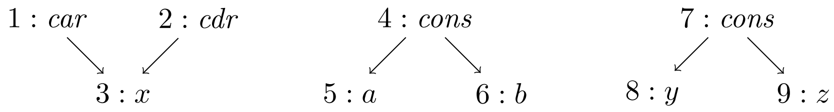


Consider $x \neq cons(y, z)$, we have `FIND` 3 $\neq$ `FIND` 4. Thus, the formula is T_{cons}-satisfiable.

(b) Preprocess the formula and get the following one:

$$x = cons(a, b) \wedge car(x) = y \wedge cdr(x) = z \wedge x \neq cons(y, z).$$

Below is the initial DAG.



(1) Add nodes 10 : $car(cons(a, b))$ and 12 : $car(cons(y, z))$, and `MERGE` 10 5 and `MERGE` 12 8 (by left projection)

(2) Add nodes 11 : $cdr(cons(a, b))$ and 13 : $cdr(cons(y, z))$, and `MERGE` 11 6 and `MERGE` 13 9 (by right projection)

(3) `MERGE` 3 4 (by $x = cons(a, b)$)

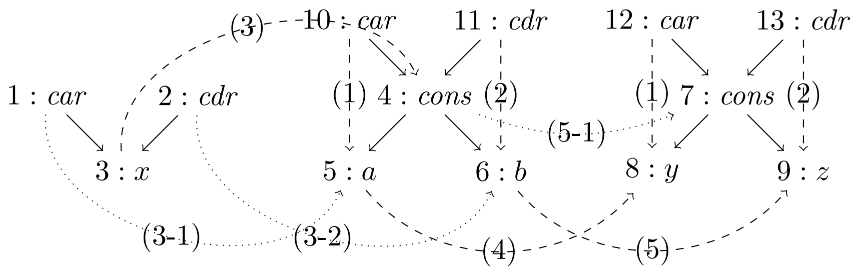
  (3-1) `MERGE` 1 10 (by function congruence)

  (3-2) `MERGE` 2 11 (by function congruence)

(4) `MERGE` 1 8 (by $car(x) = y$)

(5) `MERGE` 2 9 (by $cdr(x) = z$)

  (5-1) `MERGE` 4 7 (by function congruence)



Consider $x \neq cons(y, z)$, we have `FIND` 3 = `FIND` 7 = 7. Thus, the formula is T_{cons}-unsatisfiable.

$\square$

5. Apply the decision procedure for quantifier-free T_A to the following Σ_A-formulae.

(a) $a\langle i \triangleleft e\rangle[j] = e \wedge i \neq j$

(b) $a\langle i \triangleleft e\rangle\langle j \triangleleft f\rangle[k] = g \wedge j \neq k \wedge i = j \wedge a[k] \neq g$

Solution.

(a) Consider the following two cases.

- Case 1: $i = j$. The formula becomes

$$i = j \wedge e = e \wedge i \neq j$$

which is T_E-unsatisfiable.

- Case 2: $i \neq j$. The formula becomes

$$i \neq j \wedge a[j] = e \wedge i \neq j$$

which is T_A-satisfiable because the following formula

$$i \neq j \wedge f_a(j) = e \wedge i \neq j$$

is T_E-satisfiable.

Conclusion: T_A-satisfiable.

(b) Consider the following cases where the conversion from T_A formulas (without writing operations) to T_E formulas is applied by not shown here.

- Case 1: $j = k$. The formula becomes

$$j = k \wedge f = g \wedge j \neq k \wedge i = j \wedge a[k] \neq g$$

which is T_A-unsatisfiable.

- Case 1: $j \neq k$. The formula becomes

$$j \neq k \wedge a\langle i \triangleleft e\rangle[k] = g \wedge j \neq k \wedge i = j \wedge a[k] \neq g.$$

We have two sub-cases.

  - Case 1(a): $i = k$. The formula becomes

$$i = k \wedge j \neq k \wedge e = g \wedge j \neq k \wedge i = j \wedge a[k] \neq g$$

which is T_A-unsatisfiable.

  - Case 1(b): $i \neq k$. The formula becomes

$$i \neq k \wedge j \neq k \wedge a[k] = g \wedge j \neq k \wedge i = j \wedge a[k] \neq g$$

which is T_A-unsatisfiable.

Conclusion: T_A-unsatisfiable.

□