Elementary Computation Theory

Ming-Hsien Tsai

Institute of Information Science
Academia Sinica

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Outline

- Finite state automata
- Regular Expressions
- WS1S
- ω-Automata
- Linear temporal logic

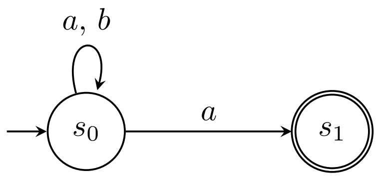
Computation

- What is the model of a computation machine?
- What is the result of a computation?

Computation

- What is the model of a computation machine?
- What is the result of a computation?
- The simplest model of computation machinery
  - *Finite state automata* (FSA), or equivalently nondeterministic finite automata (NFA), *nondeterministic finite word automata* (NFW)

Automaton M_1



- This automaton recognizes *words* (strings) end with an "a".
  - Alphabet: $\{a, b\}$
  - States: $\{s_0, s_1\}$
  - Initial states: $\{s_0\}$
  - Transitions: $\{(s_0, a, s_0), (s_0, a, s_1), (s_0, b, s_0)\}$
  - Accepting states: $\{s_1\}$

Alphabet

- An *alphabet* is a set of symbols.
- Types of alphabet: *classical* and *propositional*
- Examples:
  - $\{a, b\}$
  - $\{send, receive, ack\}$
  - $\{(p\ q), (\neg p\ q), (p\ \neg q), (\neg p\ \neg q)\}$

Words

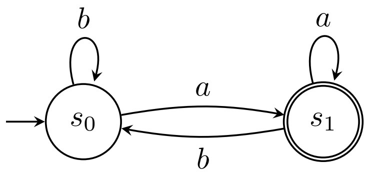
- Let Σ be a finite alphabet.
- A *word* w over Σ ($w \in \Sigma^*$) is a sequence of symbols $a_0 a_1 a_2 \ldots a_{n\text{-}1}$ with $a_i \in \Sigma$.
  - Length of w is n.
  - The empty word is denoted by ϵ.
- Examples ($\Sigma = \{a, b\}$):
  - $a\ b\ b\ a$
  - $a\ b\ a\ b\ a\ b$

w^* : repeat w finitely many times

Finite State Automata Syntax

- A finite state automaton is a 5-tuple $(Q, \Sigma, \delta, I, F)$ where
  - Q is a finite set of *states*,
  - Σ is a finite *alphabet*,
  - $\delta : Q \times \Sigma \to 2^Q$ is the *transition function* (sometimes written as a relation $\delta : Q \times \Sigma \times Q$),
  - $I \subseteq Q$ is the set of *initial states*, and
  - $F \subseteq Q$ is the set of *accepting (final) states*

Automaton M_2



$A = (Q, \Sigma, \delta, I, F)$

$\Sigma = \{a, b\}$

$Q = ?$

$I = ?$

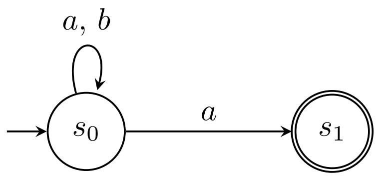
$\delta = ?$

$F = ?$

Finite State Automata Semantics

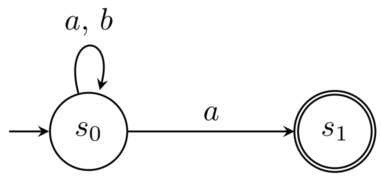
- Let $M = (Q, \Sigma, \delta, I, F)$ be a finite state automaton.
- Let $w = a_0 a_1 a_2 \ldots a_{n\text{-}1}$ be a word over Σ.
- A *run* of w on M is a sequence of states $s_0 s_1 s_2 \ldots s_n$ where
  - $s_0 \in I$
  - $(s_i, a_i, s_{i+1}) \in \delta$

Runs



- What are the runs of the following words?
  - $a\ b\ a\ b$
  - $a\ b\ b\ a$

Runs



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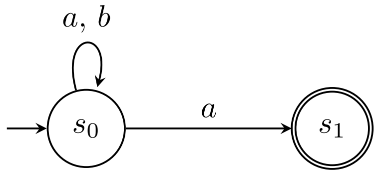
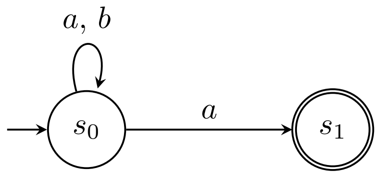
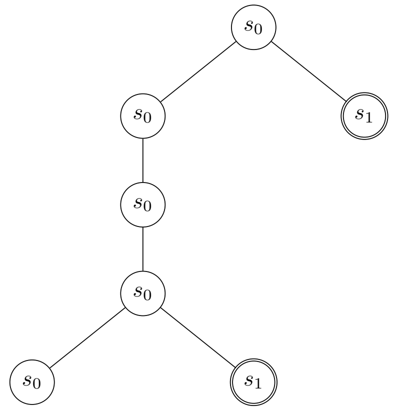
Runs



- What are the runs of the following words?
  - $a\ b\ a\ b$ $\qquad$ $s_0\ s_0\ s_0\ s_0\ s_0$
  - $a\ b\ b\ a$ $\qquad$ $s_0\ s_0\ s_0\ s_0\ s_0$ and $s_0\ s_0\ s_0\ s_0\ s_1$

Run Tree



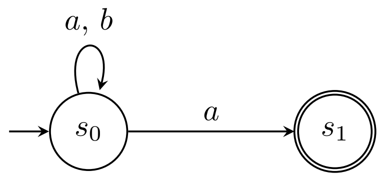


the run tree of $abba$ on M_1

Finite State Automata Semantics (cont'd)

- $M = (Q,\ \Sigma,\ \delta,\ I,\ F)$
- A run $s_0 s_1 s_2 \ldots s_n$ is *accepting* if $s_n \in F$.
- A word w is accepted by M if there is an accepting run of w on M.
- The *language* of M is the set of strings accepted by M, denoted by $L(M)$.

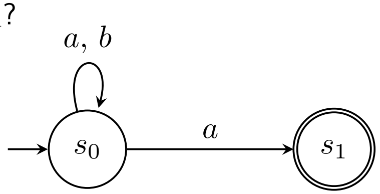
Accepting Runs



- Which run is accepting?
  - $s_0\ s_0\ s_0\ s_0\ s_0$
  - $s_0\ s_0\ s_0\ s_0\ s_1$

Languages

- What is the language of M_1?



- The language recognized by a finite state automaton is a *regular language*.

Languages

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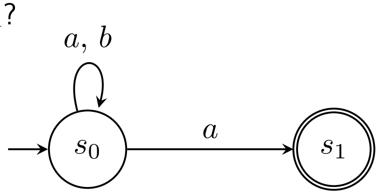


$L(M_1) = \{\ a_0a_1\ldots a_n \mid n\in\mathbb{N}$ *and* $a_n = a\}$

- The language recognized by a finite state automaton is a *regular language*.

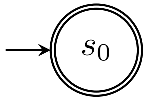
Exercise

- Given an alphabet $\{1, 2, +\}$, draw a finite state automaton such that the automaton accepts words evaluated to 3.

Emptiness and Universality

- $M = (Q, \Sigma, \delta, I, F)$
- An automaton M is *empty* if $L(M) = \varnothing$.
- An automaton M is *universal* if $L(M) = \Sigma^*$.

Emptiness and Universality



is this automaton empty?

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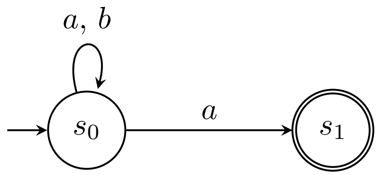
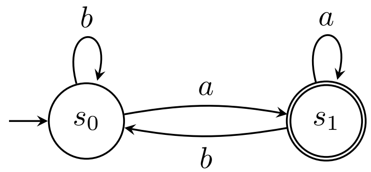
Equivalence

- Two automata are *equivalent* if they recognize the same language.





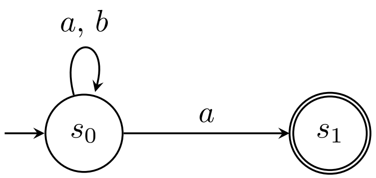
$$L(M_1) = L(M_2)?$$

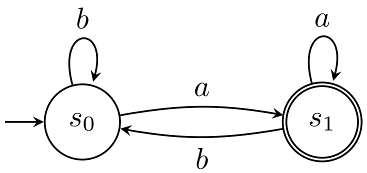
Deterministic Finite Automata (DFA)

- An automaton $M = (Q, \Sigma, \delta, I, F)$ is *deterministic* if
  - $|I| = 1$, and
  - $|\delta(s, a)| = 1$ for all $s \in Q$ and $a \in \Sigma$.

(is complete if $|\delta(s, a)| \geq 1$)

- Which one is deterministic?





Determinism VS Nondeterminism

- Let D be a DFA. The language $L(D)$ is accepted by the NFA D. (A DFA is also an NFA.)
- Let N be an NFA. Can we construct a DFA D such that $L(D) = L(N)$?

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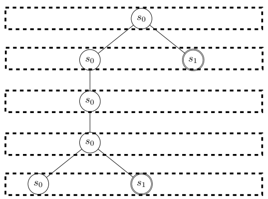
- Let D be a DFA. The language $L(D)$ is accepted by the NFA D. (A DFA is also an NFA.)
- Let N be an NFA. Can we construct a DFA D such that $L(D) = L(N)$?
- DFA and NFA have the same expressive power.

Determinization

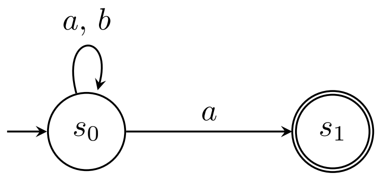
- Let $N = (Q, \Sigma, \delta, I, F)$.
- By *subset construction*, define $D = (2^Q, \Sigma, \Delta, \{ I \}, G)$ where
  - $\Delta(S, a) = \cup_{s \in S} \delta(s, a)$, and
  - $G = \{ S \in 2^Q \mid S \cap F \neq \varnothing \}$.

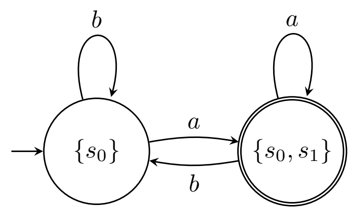


- We can show that $L(N) = L(D)$ by induction on the length of input words.

Subset Construction

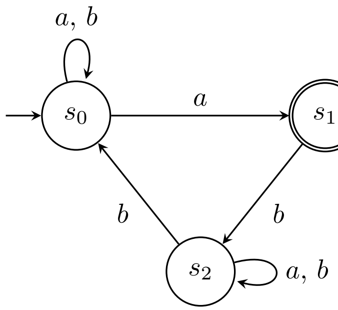
- What is the determinization of M_1?





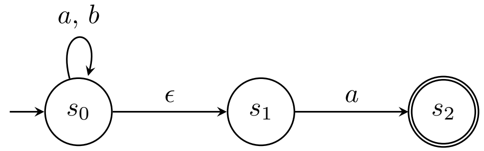
Exercise

- Apply subset construction to determinize the following automaton



ϵ-Transitions

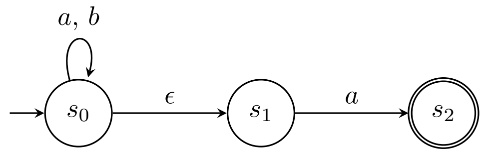
- Assume ϵ does not belong to the alphabet.
- An ϵ-transition is a transition that does not need to consume any symbol.
- ϵ-transitions are only allowed in NFA.
- DFA and NFA with ϵ-transitions have the same expressive power.



Elimination of ϵ-Transitions

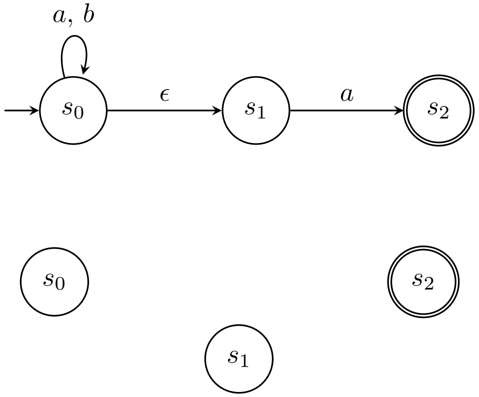
- $M = (Q, \Sigma \cup \{\epsilon\}, \delta, I, F)$ is an NFA with ϵ-transitions.
- Let $E(X)$ denote the ϵ-closure of $X \subseteq Q$.
  - $E(X) = \{\ s \mid s \in X$ *or* s *is reachable from a state in* X *through* ϵ-*transitions* $\}$
- Construct an NFA $N = (Q, \Sigma, \Delta, J, F)$ *where*
  - $\Delta(s, a) = E(\delta(s, a))$, and
  - $J = E(I)$

Elimination of ϵ-Transitions Example



Elimination of ϵ-Transitions

Example



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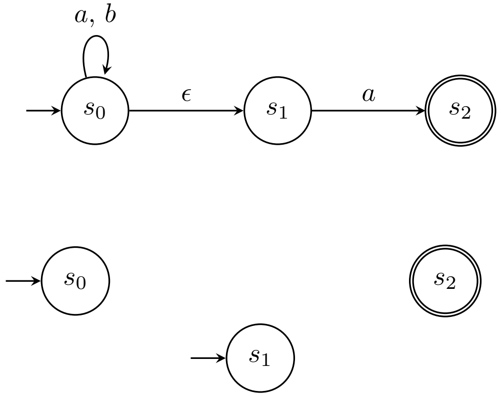

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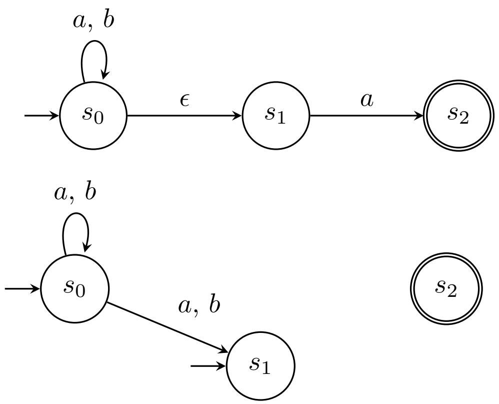

Elimination of ϵ-Transitions

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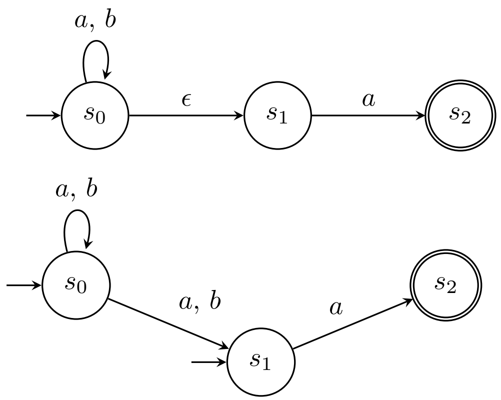
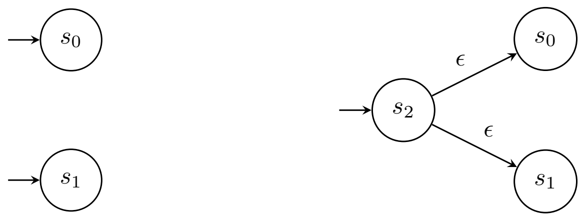


Single Initial State

- NFA may be defined as automata with single initial state.
- NFA with multiple initial states does not have more expressive power.



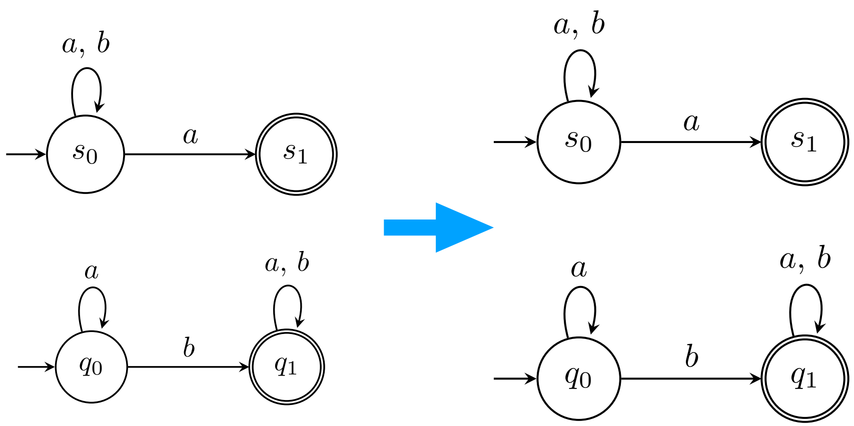
Closure Properties

- Regular languages are closed under the following operations.
  - union,
  - intersection,
  - concatenation,
  - Kleene closure, and
  - complementation.

Union

- $M_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$, $M_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$
- *Assume* $Q_1 \cap Q_2 = \varnothing$.
- $M_3 = (Q_1 \cup Q_2, \Sigma, \delta_3, I_1 \cup I_2, F_1 \cup F_2)$ where $(s, a, t) \in \delta_3$ if
  - $(s, a, t) \in \delta_1$, *or*
  - $(s, a, t) \in \delta_2$
- $L(M_3) = L(M_1) \cup L(M_2)$

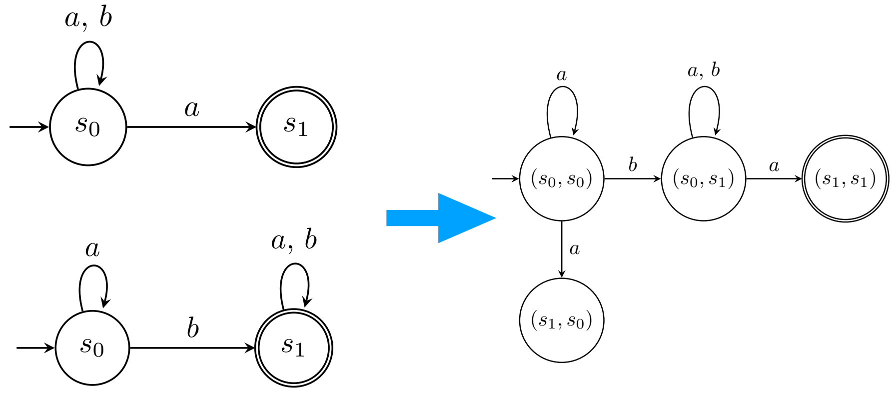
Union Example



Intersection

- $M_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$, $M_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$
- $M_3 = (Q_1 \times Q_2, \Sigma, \delta_3, I_1 \times I_2, F_1 \times F_2)$ where $((s_1, s_2), a, (t_1, t_2)) \in \delta_3$ if
  - $(s_1, a, t_1) \in \delta_1$, and
  - $(s_2, a, t_2) \in \delta_2$
- $L(M_3) = L(M_1) \cap L(M_2)$

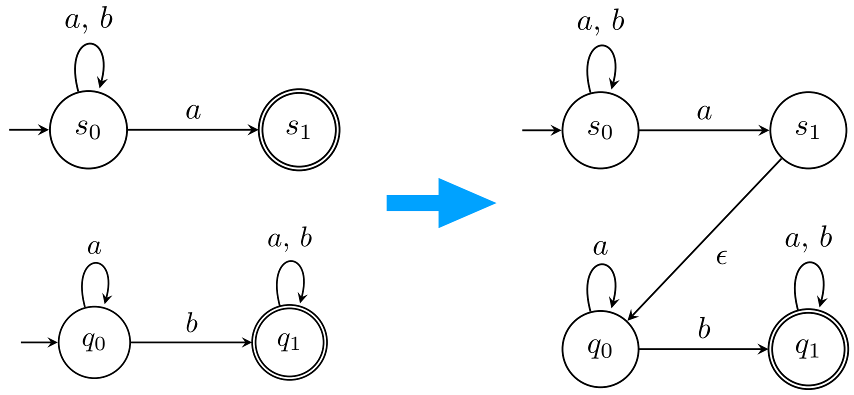
Intersection Example



Concatenation

- $M_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$, $M_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$
- *Assume* $Q_1 \cap Q_2 = \emptyset$ *and* $\epsilon \notin \Sigma$.
- $M_3 = (Q_1 \cup Q_2, \Sigma \cup \{\epsilon\}, \delta_3, I_1, F_2)$ where $(s, a, t) \in \delta_3$ if
  - $(s, a, t) \in \delta_1$,
  - $(s, a, t) \in \delta_2$, *or*
  - $a = \epsilon$, $s \in F_1$, and $t \in I_2$.
- $L(M_3) = L(M_1)L(M_2) = \{ uv \mid u \in L(M_1) \text{ and } v \in L(M_2) \}$

Concatenation Example



Kleene Closure

- An operation that repeat a string arbitrary number of times (including zero time).

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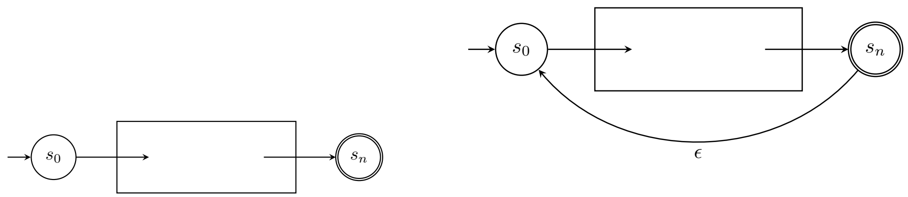

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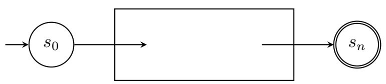
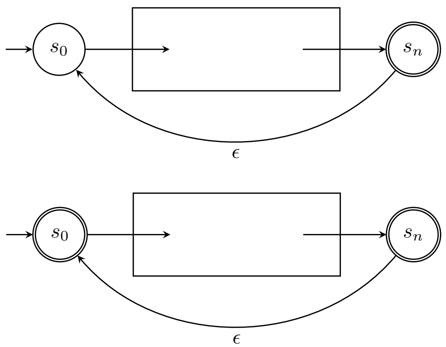
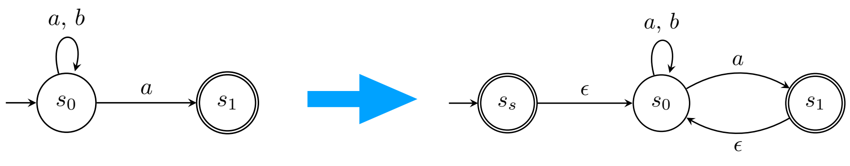




Kleene Closure (cont'd)

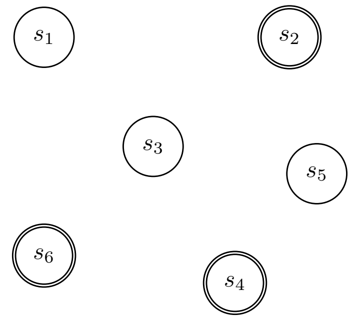
- $M = (Q, \Sigma, \delta, I, F)$
- Assume $\epsilon \notin \Sigma$ and $s_s \notin Q$.
- $M' = (Q \cup \{s_s\}, \Sigma \cup \{\epsilon\}, \Delta, \{s_s\}, F \cup \{s_s\})$ where $(s, a, t) \in \Delta$ if
  - $s = s_s$, $t \in I$, and $a = \epsilon$,
  - $(s, a, t) \in \delta$, or
  - $s \in F$, $t \in I$, and $a = \epsilon$.
- $L(M') = L(M)^*$

Kleene Closure
Example



Complementation
DFA

- $M = (Q, \Sigma, \delta, I, F)$ is a DFA.
- $M' = (Q, \Sigma, \delta, I, Q \setminus F)$
- $L(M') = \Sigma^* \setminus L(M)$



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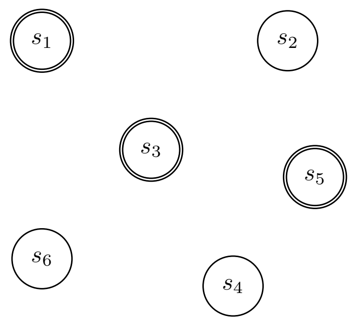

Complementation
NFA

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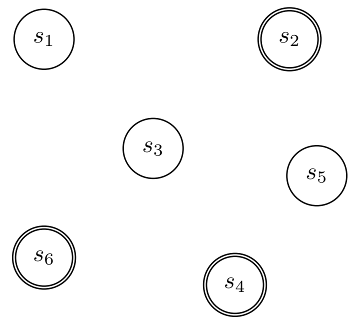

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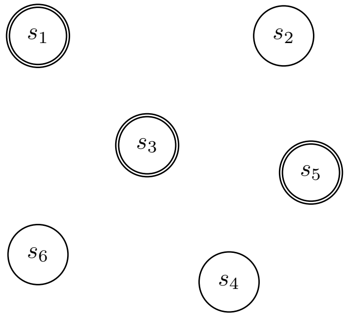

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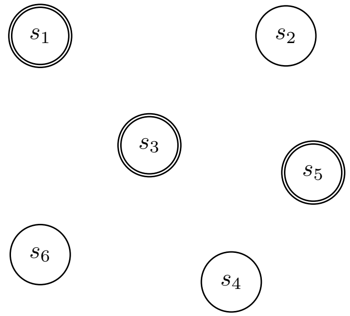

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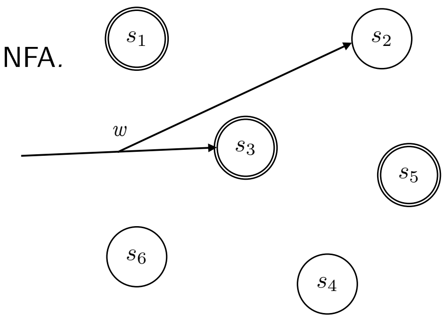

Exercise

- Let $M_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$ be two NFAs. Construct an NFA M_3 such that $L(M_3) = L(M_1) \setminus L(M_2)$. Please describe the components of M_3 in detail.

Minimization

- Given a DFA M_1, can we construct a minimal DFA M_2 such that $L(M_1) = L(M_2)$?
- Given an NFA M_1, can we construct a minimal NFA M_2 such that $L(M_1) = L(M_2)$?

Minimization

- Given a DFA M_1, can we construct a minimal DFA M_2 such that $L(M_1) = L(M_2)$? ◯
- Given an NFA M_1, can we construct a minimal NFA M_2 such that $L(M_1) = L(M_2)$?

Minimization

- Given a DFA M_1, can we construct a minimal DFA M_2 such that $L(M_1) = L(M_2)$? ◯
- Given an NFA M_1, can we construct a minimal NFA M_2 such that $L(M_1) = L(M_2)$? ◯ **but harder**

Myhill-Nerode Theorem

- Given a language $L \subseteq \Sigma^*$, define a binary relation R_L over Σ^* as follows.
  - xR_Ly *iff* $\forall z \in \Sigma^* \, (xz \in L \leftrightarrow yz \in L)$
- R_L can be shown to be an equivalence relation.
- R_L divide the set of string into *equivalence classes*.
- L is regular iff R_L has a finite number of equivalence classes.
- The number of states in the minimal DFA recognizing L is equal to the number of equivalence classes in R_L.

Minimization

Idea

- For a language $L \subseteq \Sigma^*$, compute the equivalence classes of L.
- Construct a state for each equivalence class.
- A equivalence class C_1 can take an a-transition to another equivalence class C_2 if there is a string $x \in C_1$ such that $xa \in C_2$.
- How to find the equivalence classes?

Minimization

Hopcroft's Algorithm

```
P := {F, Q \ F};
W := {F};
while (W is not empty) do
     choose and remove a set A from W
     for each c in Σ do
          let X be the set of states for which a transition on c leads to a state in A
          for each set Y in P for which X ∩ Y is nonempty and Y \ X is nonempty do
               replace Y in P by the two sets X ∩ Y and Y \ X
               if Y is in W
                    replace Y in W by the same two sets
               else
                    if |X ∩ Y| <= |Y \ X|
                         add X ∩ Y to W
                    else
                         add Y \ X to W
          end;
     end;
end;
```

the pseudocode is taken from https://en.wikipedia.org/wiki/DFA_minimization

Language Expressions

- So far we know that a regular language can be accepted by a finite state automaton.
- Can we represent a regular language in other forms?

Language Expressions

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regular expressions

Regular Expressions (RE)

- Let Σ be an alphabet.
- The regular expressions over Σ are defined as follows.
  - $\emptyset$ is a regular expression denoting the empty set;
  - ϵ is a regular expression denoting the set $\{\epsilon\}$;
  - for each $a \in \Sigma$, a is a regular expression denoting the set $\{a\}$;
  - if r and s are regular expressions denoting the sets R and S respectively, then $r{+}s$, rs, and r^* are regular expressions denoting $R{\cup}S$, RS, and R^* respectively.
- The language of a regular expression e is denoted by $L(e)$.

Regular Expressions Examples

- Let $\Sigma = \{a,\ b\}$.
- $a^*ba^* = \{w \mid w$ *has exactly a single* $b\}$
- $\Sigma^*b\Sigma^* = \{w \mid w$ *has at least one* $b\}$
- $\Sigma^*aba\Sigma^* = \{w \mid w$ *has a substring* $aba\}$
- $a{+}b{+}a\Sigma^*a{+}b\Sigma^*b = \{w \mid w$ *starts and ends with the same symbol*$\}$

Regular Expressions Examples (cont'd)

- $r{+}\emptyset = ?$
- $r{+}\epsilon = ?$
- $r\emptyset = ?$
- $r\epsilon = ?$

Regular Expressions Examples (cont'd)

- $r{+}\emptyset = ?$ $\quad r$
- $r{+}\epsilon = ?$
- $r\emptyset = ?$
- $r\epsilon = ?$

Regular Expressions Examples (cont'd)

- $r+\emptyset = ?$ $\quad r$
- $r+\epsilon = ?$ $\quad r+\epsilon$
- $r\emptyset = ?$
- $r\epsilon = ?$

Regular Expressions Examples (cont'd)

- $r+\emptyset = ?$ $\quad r$
- $r+\epsilon = ?$ $\quad r+\epsilon$
- $r\emptyset = ?$ $\quad \emptyset$
- $r\epsilon = ?$

Regular Expressions Examples (cont'd)

- $r+\emptyset = ?$ $\quad r$
- $r+\epsilon = ?$ $\quad r+\epsilon$
- $r\emptyset = ?$ $\quad \emptyset$
- $r\epsilon = ?$ $\quad r$

Exercise

- Write regular expressions to describe the following languages. $(\Sigma = \{a, b\})$
  - $\{w \mid \text{the length of } w \text{ is even}\}$
  - $\{w \mid w \text{ has at most two } b\text{'s}\}$
  - $\{w \mid \text{every } a \text{ in } w \text{ is followed by } b\}$

Regular Expressions
vs
Finite State Automata

- A language is recognized by an NFA if and only if some regular expression describes it.
- A language is regular if and only if some regular expression describes it.

From RE to NFA

$\emptyset$ ϵ a

Let A_r be an NFA recognizing the language of a regular expression r.

$r+s$: union of A_r and A_s

rs: concatenation of A_r and A_s

$r*$: the Kleene closure of A_r

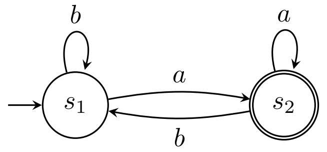
From NFA to RE

- Transitive Closure Method
- State Removal Method
- Brzozowski Algebraic Method

Transitive Closure Method

- Let $D = (\{s_1, \ldots, s_n\}, \Sigma, \delta, \{s_1\}, F)$ be a DFA.
- Define
  - $R_{ij}^{0} = \{a \mid (s_i, a, s_j) \in \delta\}$ if $i \neq j$
  - $R_{ij}^{0} = \{a \mid (s_i, a, s_j) \in \delta\} \cup \{\epsilon\}$ if $i = j$
  - $R_{ij}^{k} = R_{ik}^{k-1}(R_{kk}^{k-1})^* R_{kj}^{k-1} \cup R_{ij}^{k-1}$
- R_{ij}^{k} represents the inputs that cause D to go from s_i to s_j without passing through a state higher than s_k.
- R_{ij}^{k} can be denoted by regular expressions.
- $L(D) = \cup_{Sj \in F} R_{1j}^{n}$.

Transitive Closure Method Example



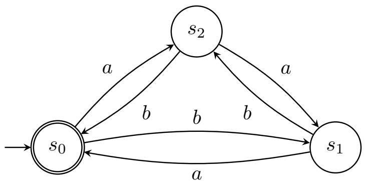
	$k=0$	$k=1$	$k=2$
${R_{11}}^k$	$b+\epsilon$	$(b+\epsilon)(b+\epsilon)^*(b+\epsilon)+(b+\epsilon)$ $= b^*$	
${R_{12}}^k$	a	$(b+\epsilon)(b+\epsilon)^*a+a$ $= b^*a$	$b^*a(b^*a+\epsilon)^*(b^*a+\epsilon)+b^*a$ $= (a+b)^*a$
${R_{21}}^k$	b	$b(b+\epsilon)^*(b+\epsilon)+b$ $= b^+$	
${R_{22}}^k$	$a+\epsilon$	$b(b+\epsilon)^*a+(a+\epsilon)$ $= b^*a+\epsilon$	

 $b^+ = bb^*$

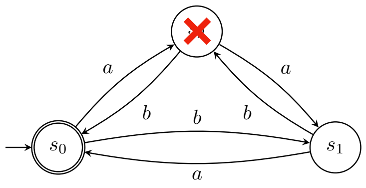
State Removal Method

- Make the NFA has a single accepting state.
- Make the NFA has a single initial state.
- Remove states and change transition labels (may be regular expressions) until there is only the initial state and the accepting state.
- Compute the regular expression.

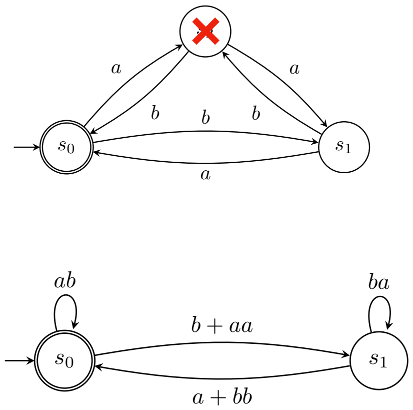
State Removal Method Example



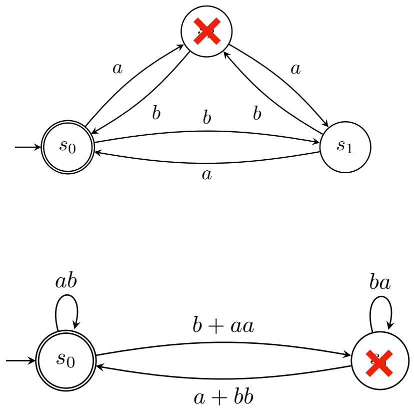
State Removal Method Example



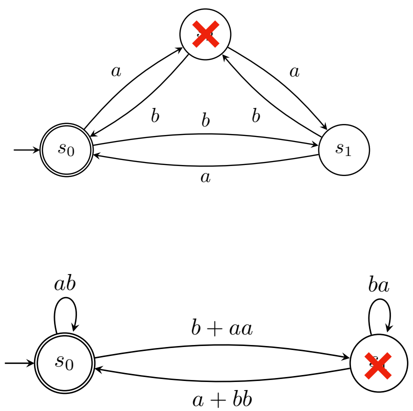
State Removal Method Example

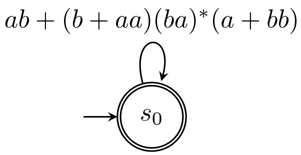


State Removal Method Example

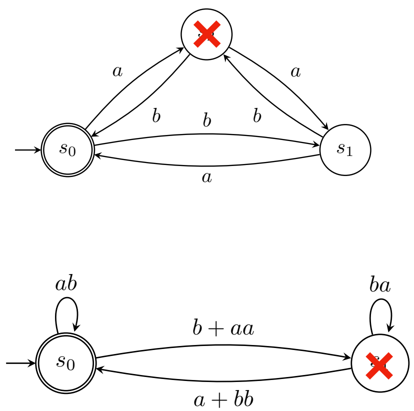


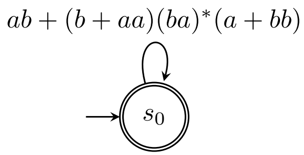
State Removal Method Example





State Removal Method Example





$$(ab+(b+aa)(ba)^*(a+bb))^*$$

Brzozowski Algebraic Method

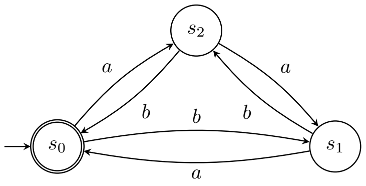
- $M = (Q, \Sigma, \delta, \{q_0\}, F)$ is an NFA containing no ϵ-transitions.

- For every q_i, create the equation

$$Q_i = +_{q_i \xrightarrow{a} q_j} aQ_j + \begin{cases} \{\epsilon\}, \text{if } q_i \in F \\ \varnothing, \text{else} \end{cases}$$

- Solve the equation system and find Q_0.

Brzozowski Algebraic Method Example



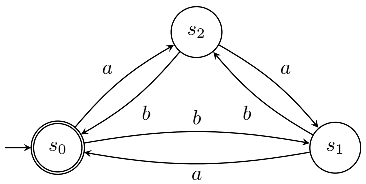
$$Q_0 = bQ_1 + aQ_2 + \epsilon$$
$$Q_1 = aQ_0 + bQ_2$$
$$Q_2 = bQ_0 + aQ_1$$

$$\begin{aligned} Q_2 &= bQ_0 + aQ_1 \\ &= bQ_0 + a(aQ_0 + bQ_2) \\ &= abQ_2 + (b+aa)Q_0 \end{aligned}$$

by Arden's Lemma:
$L = UL+V$ iff $L = U^*V$ where $L, U, V \subseteq \Sigma^*$ with $\epsilon \notin U$

$$Q_2 = (ab)^*(b+aa)Q_0$$

Brzozowski Algebraic Method Example (cont'd)



$$Q_0 = bQ_1 + aQ_2 + \epsilon$$
$$Q_1 = aQ_0 + bQ_2$$
$$Q_2 = bQ_0 + aQ_1$$

$$Q_2 = (ab)^*(b+aa)Q_0$$

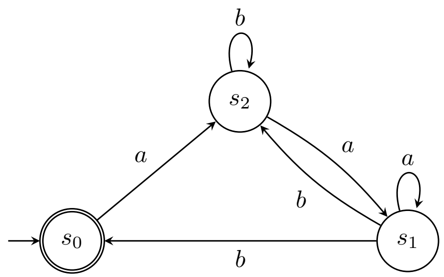
$$\begin{aligned} Q_0 &= bQ_1 + aQ_2 + \epsilon \\ &= b(aQ_0 + bQ_2) + aQ_2 + \epsilon \\ &= baQ_0 + (bb+a)Q_2 + \epsilon \\ &= (ba+(bb+a)(ab)^*(b+aa))Q_0 + \epsilon \end{aligned}$$

by Arden's Lemma:
$L = UL+V$ iff $L = U^*V$ where $L, U, V \subseteq \Sigma^*$ with $\epsilon \notin U$

$$Q_0 = (ba+(bb+a)(ab)^*(b+aa))^*$$

Exercise

- Express the language of the following automaton by a regular expression.



WS1S

- Syntax of S1S (monadic second-order logic of one successor)
  - First-order variable set: $V = \{x_1, x_2, ...\}$
  - Second-order variable set: $X = \{X_1, X_2, ...\}$
  - Terms: $t ::= 0 \mid x_i$
  - Formulas: $\varphi ::= S(t, t) \mid X_i(t) \mid \neg \varphi \mid \varphi \wedge \varphi \mid \exists x_i.\varphi \mid \exists X_i.\varphi$
- S is the successor predicate.
- WS1S: fragment of S1S which allows only quantification over finite sets

Semantics of S1S

- Signature $\langle \mathbb{N}, S \rangle$
- Interpretation $\sigma = \langle \sigma_1, \sigma_2 \rangle, \sigma_1 : V \to \mathbb{N}, \sigma_2 : X \to 2^{\mathbb{N}}$
- *Satisfiability*

$$\begin{array}{lll}
\sigma \models X(t) & \textit{iff} & \sigma(t) \in \sigma(X) \\
\sigma \models S(t, t') & \textit{iff} & \sigma(t) + 1 = \sigma(t') \\
\sigma \models \neg\varphi & \textit{iff} & \sigma \not\models \varphi \\
\sigma \models \varphi_1 \wedge \varphi_2 & \textit{iff} & \sigma \models \varphi_1 \text{ and } \sigma \models \varphi_2 \\
\sigma \models \exists x.\varphi & \textit{iff} & \sigma[n/x] \models \varphi \text{ for some } n \in \mathbb{N} \\
\sigma \models \exists X.\varphi & \textit{iff} & \sigma[N/X] \models \varphi \text{ for some } N \in 2^{\mathbb{N}}
\end{array}$$

- *Validity* $\models \varphi \quad \textit{iff} \quad \sigma \models \varphi$ for all interpretations σ

Abbreviations

$$\begin{array}{lll}
\varphi_1 \vee \varphi_2 & := & \neg(\neg\varphi_1 \wedge \neg\varphi_2) \\
\varphi_1 \to \varphi_2 & := & \neg\varphi_1 \vee \varphi_2 \\
\forall x.\varphi & := & \neg\exists x.\neg\varphi \\
\forall X.\varphi & := & \neg\exists X.\neg\varphi \\
x \leq y & := & \forall X.(y \in X \wedge \forall z.\forall z'.(z \in X \wedge S(z', z) \to z' \in X) \to X(x)) \\
x < y & := & x \leq y \wedge \neg(y \leq x) \\
\mathit{first}(x) & := & \neg\exists y.S(y, x) \\
\mathit{last}(x) & := & \neg\exists y.S(x, y) \\
X \subseteq Y & := & \forall x.(x \in X \to x \in Y) \\
X = Y & := & X \subseteq Y \wedge Y \subseteq X \\
X = \varnothing & := & \forall Z, X \subseteq Z \\
\mathit{sing}(X) & := & X \neq \varnothing \wedge \forall Y.(Y \subseteq X \to (X \subseteq Y \vee Y = \varnothing))
\end{array}$$

WS1S on Words

- Let Σ be a finite set of alphabet.
- A word is defined as $w = a_0 a_1 \ldots a_{n-1}$.
- A unary predicate P_a is defined for every $a \in \Sigma$ such that $P_a(i)$ if and only if $a_i = a$.
- Domain of w: $dom(w) = \{0, ..., |w| - 1\}$
- Word model of w: $\langle dom(w), S^w, (P_a)_{a \in \Sigma} \rangle$
- Büchi Theorem: a language $L \subseteq \Sigma^*$ is regular if and only if L is expressible in WS1S.

WS1S Examples

- the last symbol is a
  - $\exists x.(P_a(x) \wedge \neg\exists y.(x < y))$
- contains substring ab
  - $\exists x.\exists y.(P_a(x) \wedge P_b(y) \wedge S(x,y))$
- has substring ba^*b
  - $\exists x.\exists y.(x < y \wedge P_b(x) \wedge P_b(y) \wedge \forall z((x < z \wedge z < y) \rightarrow P_a(z)))$
- non-empty word with a even length
  - $\exists f.\exists l.\exists X.(\mathit{first}(f) \wedge \mathit{last}(l) \wedge X(f) \wedge \neg X(l) \wedge \forall y.\forall z.(S(y,z) \rightarrow (X(y) \leftrightarrow \neg X(z))))$

Exercises

- Write WS1S formulas to describe the following words.
  - Only a's can occur between any two occurrences of b's
  - Has an odd length (please start with $\exists$)

From NFA to WS1S

- Let $M = (Q, \Sigma, \delta, \{s_0\}, F)$ be an NFA.
- Assume $Q = \{s_0, s_1, ..., s_n\}$.
- Non-empty accepting words will satisfy the following formula.

$$
\begin{array}{lll}
\exists X_0 \ldots X_n. & (\ \wedge_{i \neq j} & \forall x.\neg(x \in X_i \wedge x \in X_j) \\
 & \wedge & \forall x.(\mathit{first}(x) \rightarrow x \in X_0) \\
 & \wedge & \forall x.\forall y.(S(x,y) \rightarrow \vee_{(s_i,a,s_j) \in \delta}(x \in X_i \wedge x \in P_a \wedge y \in X_j)) \\
 & \wedge & \forall x.(\mathit{last}(x) \rightarrow \vee_{(s_i,a,s_f) \in \delta; s_f \in F}(x \in X_i \wedge x \in P_a)))
\end{array}
$$

A Better Encoding

- Assume $|\Sigma| = 2^m$.
- A symbol is binary encoded as $(t_0, t_1, ..., t_{m\text{-}1})$.
- A word is defined as $w = a_0a_1 \ldots a_{n\text{-}1}$.
- A unary predicate P_i is defined for every $i \in \{0,...,m\text{-}1\}$ such that $P_i(j)$ if and only if the i-th track of a_j is 1.
- Example:
  - $m = 2$, $\Sigma = \{a, b, c, d\}$, $a = (00)$, $b = (01)$, $c = (10)$, $d = (11)$
  - $P_0 = \{0, 3, 4\}$, $P_1 = \{1, 4\}$
  - $w = (10)(01)(00)(10)(11) = cbacd$

Non-regular Languages

- Examples of non-regular languages:
  - $\{\ a^n b^n \mid n \in \mathbb{N}\ \}$
  - $\{\ w\#w \mid w \in \{a,\ b\}^*\ \}$
- How to prove that a language is non-regular?

Pumping Lemma

- If L is a regular language, then there is a number $p \geq 1$ (the pumping length) such that, if s is any string in L and $|s| \geq p$, then s may be divided as $s = xyz$ satisfying
  - for each $i \geq 0$, $xy^i z \in L$,
  - $|y| > 0$, and
  - $|xy| \leq p$.

Pumping Lemma Example

- Let's show that $L = \{\ a^n b^n \mid n \in \mathbb{N}\ \}$ is non-regular.
- Assume L is regular and let $w = a^p b^p$.
- By pumping lemma, there are x, y, and z such that $w = xyz$,
  - $xy^i z \in L$ for each $i \geq 0$,
  - $|y| \geq 0$, and
  - $|xy| \leq p$.
- With $|xy| \leq p$, we know that y contains only a.
- But $xy^2 z \notin L$.

Formal Languages

Chomsky Hierarchy	Grammar	Language	Computation Model
Type-0	Unrestricted	Recursively enumerable	Turing machine
Type-1	Context-sensitive	Context-sensitive	Linear-bounded
Type-2	Context-free	Context-free	Pushdown
Type-3	Regular	Regular	Finite

the list of formal languages in this table is not complete

Tools

- MONA (http://www.brics.dk/mona/)
- JFLAP (http://www.jflap.org)

Infinite Computations

- A *reactive system* is a system that continuously interacts with its environment.
- Computations of a reactive system are infinite.
- How to model such infinite computations?
  - Automata on infinite words

Infinite Words

- Let Σ be a finite alphabet.
- An infinite word *w* over Σ ($w \in \Sigma^\omega$) is a sequence of symbols $a_0 a_1 a_2 \ldots$ with $a_i \in \Sigma$.
  - Length of w is ω.
- Examples ($\Sigma = \{a, b\}$):
  - $a\ b\ (b\ a)^\omega$
  - $a\ b\ a\ (b\ a\ b)^\omega$

ω-Automata

Syntax

- An *ω-automaton* is a tuple $(Q, \Sigma, \delta, q_0, Acc)$ where
  - Q is a finite set of states,
  - Σ is a finite alphabet,
  - δ: $Q \times \Sigma \to 2^Q$ is the transition function,
  - q_0 is the initial state, and
  - Acc is the *acceptance condition*.
- Different ω-automata can be defined by different acceptance conditions.

ω-Automata

Semantics

- Let $M = (Q, \Sigma, \delta, q_0, Acc)$ be an ω-automaton.
- Let $w = a_0 a_1 a_2 \ldots$ be an infinite word over Σ.
- A *run* of w on M is a sequence of states $q_0 q_1 q_2 \ldots$ where $(q_i, a_i, q_{i+1}) \in \delta$.

ω-Automata

Semantics (cont'd)

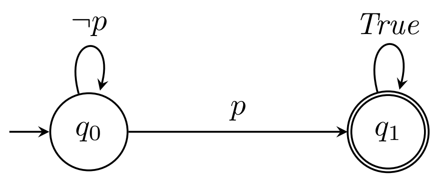
- A run is accepting if the run satisfies the acceptance condition Acc.
- A word is accepted if there is a run of M on the word.
- The language of M, denoted by $L(M)$, is the set of words accepted by M.
- Define $\mathit{Inf}(\rho) = \{s \mid s$ *occurs in* ρ *infinitely many times*$\}$.

Acceptance Conditions

Acceptance Condition	Acc	Satisfaction	Abbrev.	Note
Büchi	$Acc = F \subseteq Q$	$\mathit{Inf}(\rho) \cap F \neq \varnothing$	NBW	
co-Büchi	$Acc = F \subseteq Q$	$\mathit{Inf}(\rho) \cap F = \varnothing$	NCW	
Generalized Büchi	$Acc = \{F_1, \ldots, F_n\}$, $F_i \subseteq Q$	$\mathit{Inf}(\rho) \cap F_i \neq \varnothing$ for all $F_i \in F$	NGW	
Rabin	$Acc = \{(E_1, F_1), \ldots, (E_n, F_n)\}$, $F_i \subseteq Q$, $E_i \subseteq Q$	$\mathit{Inf}(\rho) \cap E_i = \varnothing$ and $\mathit{Inf}(\rho) \cap F_i \neq \varnothing$ for some i	NRW	
Streett	$Acc = \{(E_1, F_1), \ldots, (E_n, F_n)\}$, $F_i \subseteq Q$, $E_i \subseteq Q$	$\mathit{Inf}(\rho) \cap F_i \neq \varnothing$ implies $\mathit{Inf}(\rho) \cap E_i \neq \varnothing$ for all i	NSW	
Muller	$Acc = \{F_1, \ldots, F_n\}$, $F_i \subseteq Q$	$\mathit{Inf}(\rho) = F_i$ for some i	NMW	
Parity	$Acc: Q \to \mathbb{N}$	min parity in ρ is even	NPW	$Acc(q)$ is the parity of q

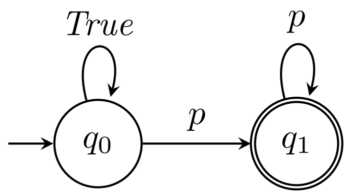
Büchi Automata

Example 1



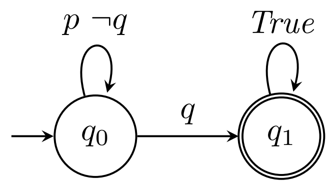
accepts infinite words where p holds eventually

Büchi Automata
Example 2



accepts infinite words where eventually p will always hold

Büchi Automata
Example 3



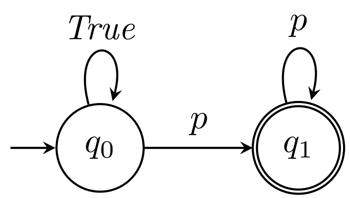
accepts infinite words where p holds until q holds

Exercise

- Draw a Büchi automaton that accepts infinite words where p holds infinitely many times. ($\Sigma = \{p, \neg p\}$)

Deterministic VS Nondeterministic

- Can you find a deterministic Büchi automaton (DBW) that accepts the same language?



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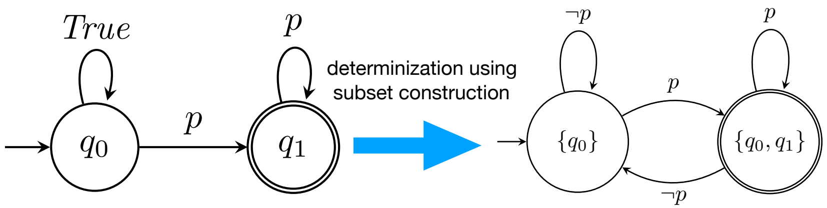

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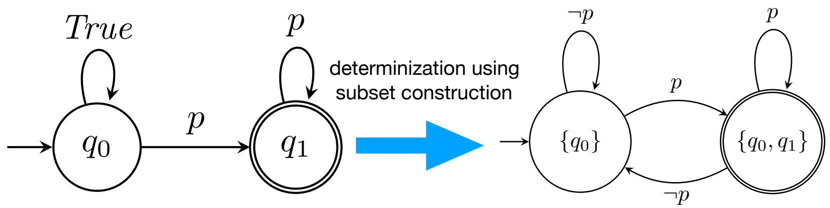
- Can you find a deterministic Büchi automaton (DBW) that accepts the same language?



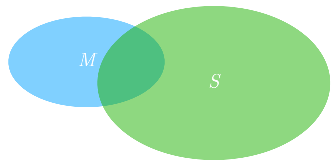
NBW is more expressive than DBW

Model VS Specification

- So far we already learnt some abstract machines as models of computations.
- We may require that the computations must satisfy some properties.
- How do we check?

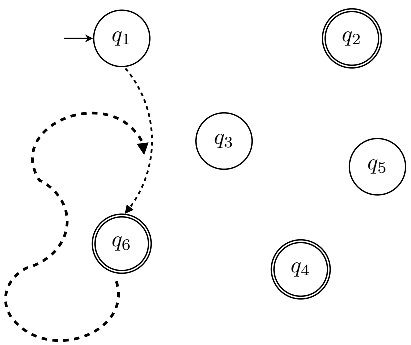
Model Checking

- Model the computations of a system as an automaton M.
- Model the computations allowed by the specification as an automaton S.
- Check if the system satisfies the specification by checking if $L(M) \subseteq L(S)$.
- Or equivalently checking if P is empty where P is the intersection of
  - M and
  - the complement of S.



Emptiness Test

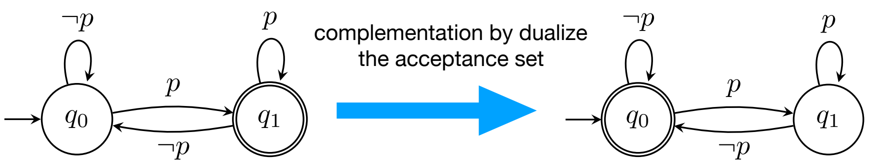
- Use double depth-first search to find an accepting lasso.



Büchi Automata Intersection

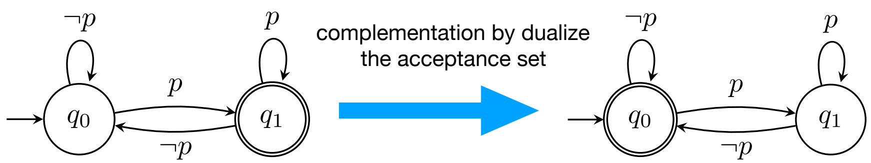
- $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$, $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$
- Construct $M = (Q_1 \times Q_2 \times \{0,1,2\}, \Sigma, \delta, (q_{01}, q_{02}, 0), Q_1 \times Q_2 \times \{0\})$ where $((q_1, q_2, i), a, (q_1', q_2', j)) \in \delta$ if
  - $(q_1, a, q_1') \in \delta_1$ and $(q_2, a, q_2') \in \delta_2$,
  - $j = 1$ if $i = 0$,
  - $j = i$ if $i \neq 0$ and $q_i \notin F_i$, and
  - $j = (i + 1)\ mod\ 2$ if $i \neq 0$ and $q_i \in F_i$.
- $L(M) = L(M_1) \cap L(M_2)$

Büchi Automata Complementation



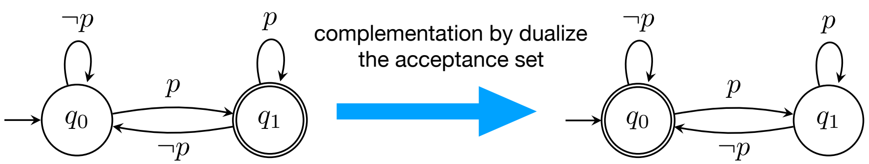
Does the right one exactly accept the complement of the left one?

Büchi Automata Complementation



Does the right one exactly accept the complement of the left one?

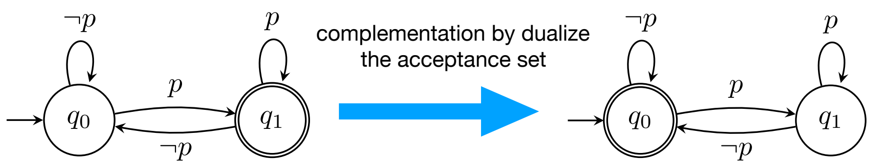
Büchi Automata Complementation



Does the right one exactly accept the complement of the left one?

Complementation of NBW is much harder than that of NFA.

Büchi Automata Complementation



Does the right one exactly accept the complement of the left one?

Complementation of NBW is much harder than that of NFA.

We may express specifications using logic formulas.

LTL Model Checking

- Express the behavior of a system as a Büchi automaton M (usually converted from a Kripke structure).
- Express the specification as a formula f in *linear temporal logic* (LTL).
- Translation $\neg f$ to a Büchi automaton $A_{\neg f}$ *with labels on states*.
- Check if $L(M) \cap L(A_{\neg f})$ is empty.

Linear Temporal Logic Syntax

- AP is a finite set of atomic propositions.
- The alphabet Σ is defined as 2^{AP}.
- A linear temporal logic (LTL) formula is defined as follows.
  - For every $p \in AP$, p is an LTL formula.
  - If f and g are LTL formulas, then so are $\neg f$, $f \wedge g$, $\boldsymbol{X}\ f$, and $f\ \boldsymbol{U}\ g$.
- $\boldsymbol{X}$ and $\boldsymbol{U}$ are (future) temporal operators.

Linear Temporal Logic Semantics

- A state is a subset of AP, containing exactly those propositions that evaluate to true in that state.
- An LTL formula is interpreted over an infinite sequence of states $\rho = s_0 s_1 \ldots$.

$(\rho,\, i) \models p$ iff $p \in s_i$

$(\rho,\, i) \models \neg f$ iff $(\rho,\, i) \not\models f$

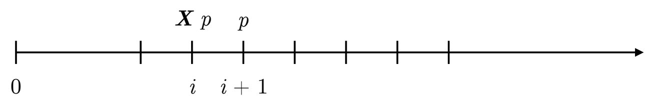
$(\rho,\, i) \models f \wedge g$ iff $(\rho,\, i) \models f$ and $(\rho,\, i) \models g$

$(\rho,\, i) \models \boldsymbol{X}\, f$ iff $(\rho,\, i+1) \models f$

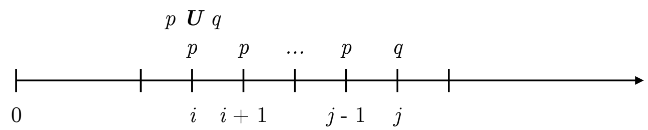
$(\rho,\, i) \models f\ \boldsymbol{U}\ g$ iff exists $j \geq i$ such that $(\rho,\, j) \models g$ and for all $i \leq k < j$, $(\rho,\, k) \models f$

Next and Until

- $(\rho,\, i) \models \boldsymbol{X}\, f$ iff $(\rho,\, i+1) \models f$

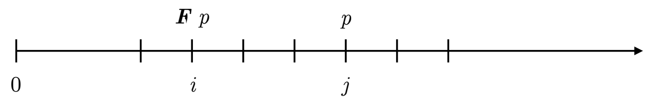


- $(\rho,\, i) \models f\ \boldsymbol{U}\ g$ iff exists $j \geq i$ such that $(\rho,\, j) \models g$ and for all $i \leq k < j$, $(\rho,\, k) \models f$

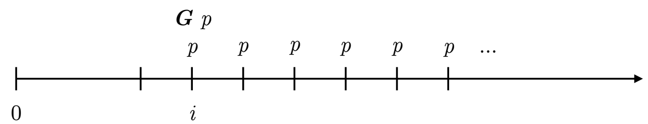


Future and Global

- $(\rho,\, i) \models \boldsymbol{F}\, f$ iff $(\rho,\, j) \models f$ for some $j \geq i$

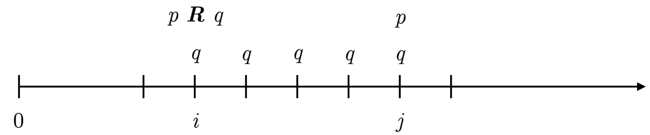


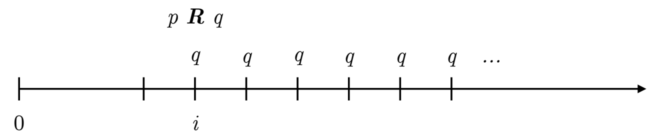
- $(\rho,\, i) \models \boldsymbol{G}\, f$ iff $(\rho,\, j) \models f$ for all $j \geq i$



Release

- $(\rho,\, i) \models f\ \boldsymbol{R}\ g$ iff exists $j \geq i$ such that $(\rho,\, j) \models f$ and for all $i \leq k \leq j$, $(\rho,\, k) \models g$; or for all $j \geq i$, $(\rho,\, j) \models g$





Abbreviations

- $true := p \vee \neg p$
- $false := \neg true$
- $f \vee g := \neg(\neg f \wedge \neg g)$
- $f \rightarrow g := \neg f \vee g$
- $f \leftrightarrow g := (f \rightarrow g) \wedge (g \rightarrow f)$
- $f\ \boldsymbol{R}\ g := \neg(\neg f\ \boldsymbol{U}\ \neg g)$
- $\boldsymbol{F}\ g := true\ \boldsymbol{U}\ g$
- $\boldsymbol{G}\ f := false\ \boldsymbol{R}\ f$

$\bigcirc = \boldsymbol{X}$, $\Diamond = \boldsymbol{F}$, $\Box = \boldsymbol{G}$

Exercise

- Express the following sentences in LTL formulas.
  - "p occurs infinitely often"
  - "whenever a message is sent, eventually an acknowledgement will be received"

Satisfaction, Validity, and Congruence

- $\rho \vDash f$: a state sequence ρ *satisfies* an LTL formula f
  - $\rho \vDash f$ iff $(\rho, 0) \vDash f$
- $\vDash f$: an LTL formula f is *valid*
  - $\vDash f$ iff $\rho \vDash f$ for all ρ
- $f \cong g$: two formulas f and g are *congruent*
  - $f \cong g$ iff $\vDash \boldsymbol{G}\ (f \leftrightarrow g)$

Congruent Formulas

- $\neg\ \boldsymbol{X}\ f \cong \boldsymbol{X}\ \neg\ f$
- $\neg\ \boldsymbol{F}\ g \cong \boldsymbol{G}\ \neg\ g$
- $\neg\ \boldsymbol{G}\ f \cong \boldsymbol{F}\ \neg\ f$
- $\boldsymbol{G}\ \boldsymbol{G}\ f \cong \boldsymbol{G}\ f$
- $\boldsymbol{F}\ \boldsymbol{F}\ g \cong \boldsymbol{F}\ g$
- $\neg\ \neg\ f \cong f$

Basic Formulas

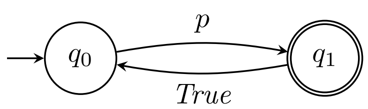
- A *literal* is either a proposition or its negation.
- A basic formula is either a literal or an $\boldsymbol{X}$-formula.

Expansion Formulas

- $\boldsymbol{F}\ g \cong g \vee \boldsymbol{X}\ \boldsymbol{F}\ g$
- $\boldsymbol{G}\ f \cong f \wedge \boldsymbol{X}\ \boldsymbol{G}\ f$
- $f\ \boldsymbol{U}\ g \cong g \vee (f \wedge \boldsymbol{X}\ (f\ \boldsymbol{U}\ g))$
- $f\ \boldsymbol{R}\ g \cong g \wedge (f \vee \boldsymbol{X}\ (f\ \boldsymbol{R}\ g))$

Expressive Power of LTL

- LTL is strictly less expressive than NBW.
- "even p" can be expressed in NBW but not LTL.



- NBW is as expressive as QPTL (Quantified Propositional Temporal Logic).
- "even p" in QPTL: $\exists\ t.\ t \wedge \boldsymbol{G}\ (t \leftrightarrow \boldsymbol{X}\ \neg t) \wedge \boldsymbol{G}\ (t \rightarrow p)$

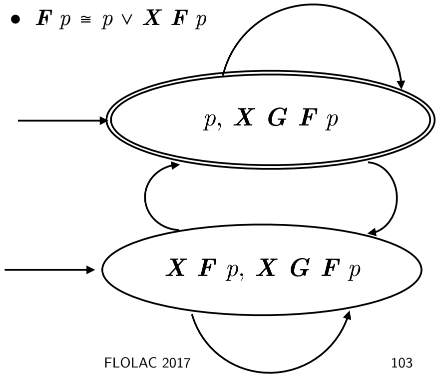
From LTL to Labeled NGW

- Translate an LTL formula f to a labeled NGW (with labels on states).
  - Take the *negation normal form* (NNF) of f.
  - Expand f_{NNF} into basic formulas as the initial states.
  - Construct successors of states based on $\boldsymbol{X}$-formulas.
  - For each subformula $g\ \boldsymbol{U}\ h$, create an acceptance set such that h will become true eventually.

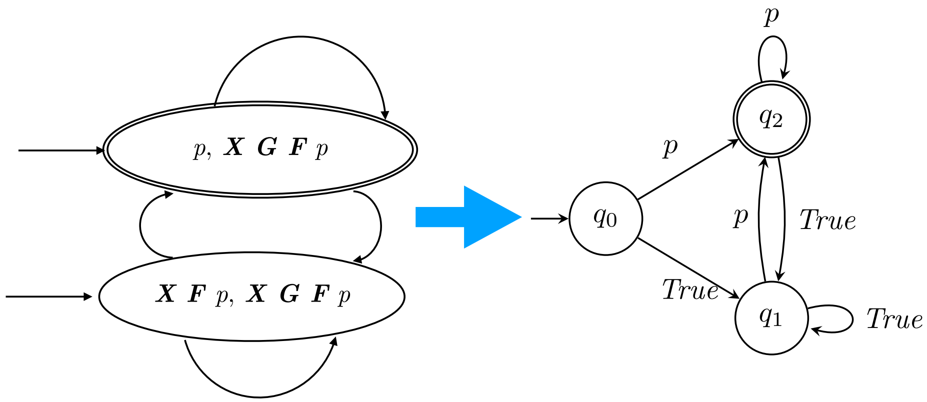
NNF: negation only occurs right before propositions

From LTL to Labeled NGW Example

- $f := \boldsymbol{G}\ \boldsymbol{F}\ p$
- $\boldsymbol{G}\ \boldsymbol{F}\ p \cong (p \vee \boldsymbol{X}\ \boldsymbol{F}\ p) \wedge \boldsymbol{X}\ \boldsymbol{G}\ \boldsymbol{F}\ p \cong (p \wedge \boldsymbol{X}\ \boldsymbol{G}\ \boldsymbol{F}\ p) \vee (\boldsymbol{X}\ \boldsymbol{F}\ p \wedge \boldsymbol{X}\ \boldsymbol{G}\ \boldsymbol{F}\ p)$
- $\boldsymbol{F}\ p \cong p \vee \boldsymbol{X}\ \boldsymbol{F}\ p$



From Labeled NGW to NGW



From NGW to NBW

- Apply the same technique in the intersection of NBW.
- Use an index i to remember the next acceptance set in $\{F_1, F_2, ..., F_n\}$ to be passed.
- Once a state in F_i is passed, increase the index i by 1.
- If every $F_i \in \{F_1, F_2, ..., F_n\}$ has been passed at least once, change the index to 0 and set the index to 1 in the successors.
- A run is accepting if the index 0 is passed infinitely many times.

Tools

- LTL2BA (http://www.lsv.fr/~gastin/ltl2ba/index.php)
- LTL3BA (https://sourceforge.net/projects/ltl3ba/)
- SPIN (http://spinroot.com/spin/whatispin.html)
- NuSMV (http://nusmv.fbk.eu)
- GOAL (http://goal.im.ntu.edu.tw/wiki/doku.php)