PROPOSITIONAL LOGIC (SESSION 1)

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WHY STUDY FORMAL LOGIC?

- Aid reasoning
- Remove ambiguity in natural language
- Mechanise reasoning
- Deep connection with computation

WHY STUDY PROPOSITIONAL LOGIC?

- The simplest, yet the most useful, formal logic
- One of the oldest formal logic (from 300 BC)
- Ubiquitous in computer science

OUR GOAL TODAY

- Introduction (reminder?) to propositional logic
- Be familiar with fundamental concepts, e.g.,:
  - syntax and semantics
  - satisfiability vs. validity
  - proofs
  - normal forms
- Getting started with SAT-solvers
- Solving interesting problems with SAT

FOOD FOR THOUGHT

How does logic relate to computation?

How does logic relate to programming?

SYNTAX (APPEARANCE) VS. SEMANTICS (MEANING)

PROPOSITIONAL LOGIC: SYNTAX OF FORMULAS

- (Atomic) Proposition (a.k.a. variable): $P, Q, \ldots$

  e.g. $P = \text{"It rains"}$ $\quad Q = \text{"I am wet"}$

- (Logical) Connectives: $\wedge, \vee, \leftrightarrow, \rightarrow, \neg, \oplus$

  e.g. $P \rightarrow Q$

  e.g. $(p \vee (q \rightarrow \neg a)) \wedge (\neg p \vee a \vee \neg b)$

CONNECTIVES (OPERATORS)

$\wedge, \vee, \leftrightarrow, \rightarrow, \oplus$ are binary operators

$\neg$ is a unary operator

The names are:

- AND: $\wedge$ （和） *a.k.a. conjunction*
- OR: $\vee$ （或） *a.k.a. disjunction*
- IMPLIES （If X, then Y): $\longrightarrow$ （若 *X* 則 *Y)*
- IF AND ONLY IF (IFF): $\leftrightarrow$ （若且唯若）
- EXCLUSIVE OR (XOR): $\oplus$

WARNING:
SO FAR, FORMULAS ARE JUST A BUNCH OF SYMBOLS WITH NO "MEANINGS"

PROPOSITIONAL LOGIC: SEMANTICS OF FORMULAS

Goal: assigning "meanings" to formulas

No grey area: a formula can only be *100% true* or *100% false*!

a.k.a. (truth) assignment

An interpretation I is a function mapping each proposition to either *1* (True) or *0* (False)

Logicians often write $I \models F$ (read: I satisfies F) if I makes the formula F true (*defined by induction on* F)

EXTENDING THE SEMANTICS TO GENERAL FORMULAS

Enumerate all the cases using a truth table

$I(A)$	$I(B)$	$I(A \wedge B)$
0	0	0
0	1	0
1	0	0
1	1	1

Note: Sometimes people omit mention of I

Example: A =“I ate today”, B = “I ate yesterday”

TASK:

WRITE A TRUTH TABLE FOR EACH OF THE OTHER CONNECTIVES

ANSWER:

CHECK ONLINE/TEXTBOOK (DO IT NOW IF YOU HAVEN'T!!)

COMMON PITFALLS

- OR in logic is not necessarily exclusive, unlike daily usage, e.g.,

He will join us, or he will die.
Darth Vader (talking about Luke Skywalker),
Star Wars: Emperor Strikes Back.

- Fifty shades of natural languages (e.g. THEN):
    1. Past time: *I was eating <u>then</u> so couldn't answer*
    2. Sequence: *Finish homework, <u>then</u> play*
    3. Logical inference: *If it rains, <u>then</u> I'll be wet*
    4. In addition: *I moved to Taipei because I like the city, and <u>then</u> there's so many other contributing factors.*

COMMON PITFALLS (CONT.)

- $P \to Q$ can be "vacuously" true when *P* is false

  P = "馬英九 is British"

  Q = "馬英九 is European"

PARSING AMBIGUITY

Question: Does $A \wedge B \vee C$ mean $(A \wedge B) \vee C$ or $A \wedge (B \vee C)$?

Rule 1: Always bracket your formulas to prevent parsing ambiguity

Rule 2: Avoid unnecessary bracketing, e.g., never write:

$$(((A) \wedge (B \vee C)))$$

TASK:

WRITE A TRUTH TABLE FOR:

1. $(A \wedge B) \vee C$
2. $A \wedge (B \vee C)$

LOOK DIFFERENT BUT MEAN THE SAME

(SEMANTICAL) EQUIVALENCE

Two formulas *F* and *F'* are (semantically) equivalent (write $F \equiv F'$) if they agree on every interpretation, i.e.,

$$\boxed{\text{For each } I,\ I(F) = I(F')}$$

Exercises: check whether the following equivalences hold

1. $(A \wedge B) \vee C \equiv A \wedge (B \vee C)$
2. $A \wedge B \equiv B \wedge A$
3. $A \equiv \neg\neg A$

WORKED-OUT EXAMPLE 1

Task: *Prove that* $A \rightarrow B \equiv \neg A \vee B$

Solution:

A	B	$\neg A$	$A \rightarrow B$	$\neg A \vee B$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

These values agree!

WORKED-OUT EXAMPLE 2

Task: *Disprove the equivalence*

$$(A \wedge B) \vee C \equiv A \wedge (B \vee C)$$

Solution: *Only need to find one interpretation where the two sides disagree, e.g.,*

$$A \mapsto 0, B \mapsto 0, C \mapsto 1$$

MORE EQUIVALENT FORMULAS

IFF = If and only if

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

Two De-Morgan's Laws:

Exercise:

Check these!!

$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$

$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

Distributivity:

$$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$$

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

MORE EXERCISES

1. Express the all-true formula $\top$ in terms of $\neg, \vee$
2. Express the all-false formula $\perp$ in terms of $\neg, \wedge$
3. Prove that $\wedge$ is *commutative* and *associative*

$$A \wedge B \equiv B \wedge A$$

$$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$$

4. Prove the same as (3) holds for $\vee$
5. Prove "Identity Laws": $A \wedge \top = A$

$$A \vee \perp = A$$

MORE EXERCISES

6. Prove "Domination Laws": $A \wedge \bot = \bot$

   $A \vee \top = \top$

7. Prove "Idempotent Laws": $A \wedge A = A$

   $A \vee A = A$

8. Prove "Contraposition": $A \to B \equiv \neg B \to \neg A$

MORE EXERCISE

Prove "Modus Ponens":

$$(A \to B) \wedge A \equiv A \wedge B$$

e.g. If I eat, then I don't starve. I eat. Therefore, I don't starve

Prove "Modus Tollens":

$$(A \to B) \wedge \neg B \equiv \neg A \wedge \neg B$$

SUBSTITUTION PRINCIPLE

Motivation: We have proven that $A \to B \equiv \neg A \vee B$

Q1: Can we deduce a similar equivalence?

$$(A \wedge C) \to B \equiv \neg(A \wedge C) \vee B$$

Q2: What about the following?

$$(A \wedge C) \to B \equiv (\neg A \vee \neg C) \vee B$$

YES! No need to reprove. Just use the substitution principle

SUBSTITUTION PRINCIPLE (CONT)

A variable substitution is a function σ mapping variables to formulas, e.g.,

$$A \mapsto (A \wedge C)$$

Extend this to all formulas by applying to each occurrence of a variable, e.g.,

$$\sigma(A \to B) = (A \wedge C) \to B$$

SUBSTITUTION PRINCIPLE (CONT)

THEOREM: Given two equivalent formulas F, F' and two variable substitution σ, σ' , if for each variable A

$$\sigma(A) \equiv \sigma'(A)$$

then

$$\sigma(F) \equiv \sigma'(F')$$

<u>Note</u>: People often use this theorem without even knowing

EXAMPLE

To prove that $(A \wedge C) \to B \equiv (\neg A \vee \neg C) \vee B$,

Step 1: $A \to B \equiv \neg A \vee B$ (shown before)

Step 2: $(A \wedge C) \to B \equiv \neg(A \wedge C) \vee B$

using substitution principle ($\sigma = \sigma' : A \mapsto A \wedge C$)

Step 3: $\neg(A \wedge C) \equiv \neg A \vee \neg C$ (De Morgan's)

Step 4: $\neg(A \wedge C) \vee B \equiv (\neg A \vee \neg C) \vee B$ (sub. pr.)

COROLLARIES OF SUBSTITUTION PRINCIPLES

All basic equivalences generalise to formulas, e.g.,

$$A \leftrightarrow B \equiv (A \to B) \land (B \to A)$$

$$\neg(A \lor B) \equiv \neg A \land \neg B$$

$$\neg(A \land B) \equiv \neg A \lor \neg B$$

$$A \land (B \lor C) \equiv (A \land B) \lor (A \land C)$$

$$A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)$$

where *A*, *B*, and *C* are formulas

EXERCISE

Prove that all propositional logic formulas can be expressed in terms of the following operators:

1. $\neg, \wedge$
2. $\neg, \vee$
3. $\neg, \rightarrow$

Note: make your use of substitution principles explicit

EXERCISE (HARDER)

Q1: Provide a recursive definition of propositional formulas

Q2: Prove substitution principle by induction

SATISFIABILITY, VALIDITY, AND *ALL THAT*

MOTIVATION

- Satisfiability and validity are fundamental concepts
- Especially useful for applications in computer science (e.g. constraint programming, knowledge reasoning, databases, verification, …)

SATISFIABILITY VS. VALIDITY

A formula is:

- satisfiable if it is true under some interpretation, eg

  “I ate today” and “I ate yesterday”

- valid if it is true under all interpretations, eg

$$((\mathrm{Eat} \to \neg\mathrm{Starve}) \wedge \mathrm{Eat}) \to \neg\mathrm{Starve}$$

- unsatisfiable if it is not satisfiable, eg

  “I cannot do it …” but actually “I can”

PONDERABLES

Q1: Can a formula and its negation be both satisfiable?

A1: YES!

Q2: What is the relationship between satisfiability and validity?

A2: A formula is valid iff its negation is unsatisfiable

MORE PONDERABLES

Q3: What is the connection between validity and equivalence?

A3: $F \equiv F'$ iff the formula $F \leftrightarrow F'$ is valid

EXERCISES

Q1: Why is the following formula valid?

$$((\mathrm{Eat} \rightarrow \neg\mathrm{Starve}) \wedge \mathrm{Eat}) \rightarrow \neg\mathrm{Starve}$$

Q2: Prove satisfiability, and disprove validity of:

$$((\mathrm{Eat} \rightarrow \neg\mathrm{Starve}) \wedge \neg\mathrm{Starve}) \rightarrow \mathrm{Eat}$$

Q3: Formalise and prove validity of: *"If Eric studies, he does not fail exams. If Eric does not play too often, he studies. Eric fails exams. Thus, Eric plays too often."*

TECHNIQUES FOR CHECKING SATISFIABILITY/VALIDITY

1. Truth table. Exponential-time (in # variables)
2. Sequent calculus (using equivalences). Can be faster, but slow in the worst case.
3. Resolution: A bit similar to sequent calculus
4. DPLL: Fast in practice (Ric will cover it)