NORMAL FORMS (SESSION 3)

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WHAT ARE NORMAL FORMS?

<u>An example from what you've learnt at high school</u>

Why do people prefer to write a polynomial as:

$$ax^2 + bx + c$$

not as:

$$7(x + x^2(x^7 + x^2 + 1) + x^3) + x^9$$

?

<u>Answer 1</u>: one looks simpler than the other

<u>Answer 2</u>: easier to compare two polynomials

<u>Answer 3</u>: ??

NORMAL FORMS FOR BOOLEAN FORMULAS

- Negation Normal Form (NNF)

$$(p \vee (q \wedge \neg a)) \wedge (\neg p \vee a \vee \neg b)$$

- Conjunctive Normal Form (CNF)

$$(p \vee \neg q) \wedge (\neg a \vee p)$$

- Disjunctive Normal Form (DNF)

$$(a \wedge \neg b \wedge c) \vee (\neg a \wedge b)$$

NEGATION NORMAL FORM (NNF)

Defn: use only $\wedge, \vee, \neg$ and $\neg$ only occurs next to atomic propositions

Example: 1. $\neg a$ 2. $(a \wedge \neg b)$

3. $(p \vee (q \wedge \neg a)) \wedge (\neg p \vee a \vee \neg b)$

Non-example:

$$\neg(\neg(a \wedge \neg b) \vee \neg(a \wedge b))$$

CONVERTING A FORMULA TO EQUIVALENT ONE IN NNF

Use the following:

1. Express the formula only in terms of

$$\wedge, \vee, \neg$$

(n.b.: this was in the exercises)

2. Push the negations "inside" using:

$$\neg(F \wedge F') \equiv \neg F \vee \neg F'$$

$$\neg(F \vee F') \equiv \neg F \wedge \neg F'$$

$$\neg\neg F \equiv F$$

EXAMPLE

Convert this to NNF:

$$\neg(\neg(a \wedge \neg b) \vee (a \wedge b))$$

Solution:

$$\neg(\neg(a \wedge \neg b) \vee (a \wedge b))$$

$$\equiv \neg\neg(a \wedge \neg b) \wedge \neg(a \wedge b) \quad \text{(De Morgan)}$$

$$\equiv (a \wedge \neg b) \wedge \neg(a \wedge b) \quad \text{(Double Negation)}$$

$$\equiv (a \wedge \neg b) \wedge (\neg a \vee \neg b) \quad \text{(De Morgan)}$$

EXERCISES

Convert the following to NNF:

1. $(a \vee \neg\neg b) \wedge (\neg c \to d)$
2. $(a \wedge \neg(\neg b \wedge c)) \wedge \neg(b \vee \neg a)$

CONJUNCTIVE NORMAL FORM (CNF)

Defn: Conjunctions of disjunctions of literals

(literal = proposition or its negation)

$$(p \vee \neg q) \wedge (\neg a \vee p)$$

Observe: CNF => NNF

Non-Example:

$$(p \vee (q \wedge \neg a)) \wedge (\neg p \vee a \vee \neg b)$$

CONVERTING A FORMULA TO EQUIVALENT ONE IN CNF

Use the following:

1. Express the formula in NNF
2. Apply the following distributivity rules:

$$F_1 \vee (F_2 \wedge F_3) \equiv (F_1 \vee F_2) \wedge (F_1 \vee F_3)$$

$$(F_1 \wedge F_2) \vee F_3 \equiv (F_1 \vee F_3) \wedge (F_2 \vee F_3)$$

i.e. push disjunctions "inside"

EXAMPLE

Convert this to CNF:

$$\neg(\neg(a \vee \neg b) \wedge \neg(a \vee b))$$

<u>Solution</u>:

1. Convert to NNF:

$$(a \vee \neg b) \vee (\neg a \wedge \neg b)$$

2. Push disjunctions inside:

$$(a \vee \neg b) \vee (\neg a \wedge \neg b)$$

$$F_1 \qquad F_2 \qquad F_3$$

$$\equiv ((a \vee \neg b) \vee \neg a) \wedge ((a \vee \neg b) \vee \neg b)$$

EXERCISES

Convert the following to CNF:

1. $(p \vee (q \wedge \neg a)) \wedge (\neg p \vee a \vee \neg b)$
2. $(a \wedge \neg\neg b) \vee (\neg c \wedge d)$

DISJUNCTIVE NORMAL FORM (DNF)

Defn: Disjunctions of conjunctions of literals

$$(a \land \neg b) \lor (\neg a \land b \land \neg c)$$

Observe: DNF => NNF

Non-Example:

$$(p \lor (q \land \neg a)) \land (\neg p \lor a \lor \neg b)$$

CONVERTING A FORMULA TO EQUIVALENT ONE IN DNF

Use the following:

1. Express the formula in NNF
2. Apply the following distributivity rules:

$$F_1 \wedge (F_2 \vee F_3) \equiv (F_1 \wedge F_2) \vee (F_1 \wedge F_3)$$

$$(F_1 \vee F_2) \wedge F_3 \equiv (F_1 \wedge F_3) \vee (F_2 \wedge F_3)$$

i.e. push conjunctions "inside"

EXERCISES

Convert the following to DNF:

1. $(p \to q) \land (q \lor \neg a)$
2. $(p \lor (q \land \neg a)) \land (\neg p \lor a \lor \neg b)$

PONDERABLES

1. Are the following formulas in CNF or DNF?

$$a \wedge \neg c$$

$$a \vee \neg c$$

2. What's the size of the formula in CNF/DNF after conversion?

3. Relationship between CNF/DNF?

PONDERABLES AT HOME

Can you give a fast algorithm for checking satisfiability of:

1. General propositional formulas
2. Formulas in NNF
3. Formulas in CNF
4. Formulas in DNF