Quantifier-Free Equality and Data Structures

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Reference book: Aaron R. Bradley and Zohar Manna. 2007. The Calculus of Computation: Decision Procedures with Applications to Verification. Springer-Verlag New York, Inc., Secaucus, NJ, USA.

Theory of Equality

- Denoted by T_E
- Referred to as the theory of EUF (Equality with Uninterpreted Functions)
- Play a central role in combining theories that share the equality predicate

In This Lecture

- The theory T_E and its quantifier-free fragment
- Deciding T_E-satisfiability of quantifier-free Σ_E-formulae
  - Congruence closure algorithm
- Implementation of the decision procedure
- T_{RDS} - recursive data structures
  - T_{cons} - lists
- T_A - arrays

Relations

Binary Relation

- Consider a set S and a binary relation R over S
- For two elements $s_1,\ s_2 \in\ S$, either s_1Rs_2 or $\neg(s_1Rs_2)$

S: Integers
R: $<$

S: Humans
R: IsChildOf

Equivalence Relation

- The relation R is an equivalence relation if it is
  - reflexive: $\forall s \in S.\ sRs$;
  - symmetric: $\forall s_1, s_2 \in S.\ s_1 R s_2 \rightarrow s_2 R s_1$;
  - transitive: $\forall s_1, s_2, s_3 \in S.\ s_1 R s_2 \wedge s_2 R s_3 \rightarrow s_1 R s_3$

$$=,\quad \bullet \equiv \bullet \pmod{c}$$

Congruence Relation

- The relation R is a congruence relation if it additionally obeys congruence: for every n-ary function f,

$$\forall S, T.\ (\wedge_{i=1 \text{ to } n}\ s_i R t_i) \rightarrow f(S) R f(T)$$

Capital S and T are vectors of variables

Theory of Equality

Signature of T_E

$$\Sigma_E: \{=,\ a,\ b,\ c,\ ...,\ f,\ g,\ h,\ ...,\ p,\ q,\ r,\ ...\},$$

consists of

- $=$, a binary predicate;
- and all constant, function and predicate symbols

Σ_E-formulae

- $x = g(y, x) \rightarrow f(x) = f(g(y, z))$
- $f(f(f(a))) = a \wedge f(f(f(f(f(a))))) = a \wedge f(a) \neq a$

$f(a) \neq a$ abbreviates $\neg(f(a) = a)$

Axioms of Equality

- Reflexivity
  - $\forall x.\ x = x$
- Symmetry
  - $\forall x, y.\ x = y \rightarrow y = x$
- Transitivity
  - $\forall x, y, z.\ x = y \wedge y = z \rightarrow x = z$

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with the three axioms, $=$ is defined to be an equivalence relation

Equality of Function Terms

- When two function terms are equal?

$$f(x) = f(g(y,\ z))$$

Function Congruence

- Function congruence (axiom schema)
  - $\forall X, Y.\ (\wedge_{i=1 \text{ to } n}\ x_i = y_i) \rightarrow f(X) = f(Y)$
- Instantiated axioms:
  - $\forall x, y.\ x = y \rightarrow f(x) = f(y)$
  - $\forall x_1, x_2, y_1, y_2.\ x_1 = y_1 \wedge x_2 = y_2 \rightarrow g(x_1, x_2) = g(y_1, y_2)$

Capital X and Y are vectors of variables

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makes $=$ a congruence relation

Capital X and Y are vectors of variables

Predicate Congruence

- Predicate congruence
    - $\forall X, Y.\ (\wedge_{i=1 \text{ to } n}\ x_i = y_i) \rightarrow (p(X) \leftrightarrow p(Y))$

T_E-Satisfiablility - Example 1

- Is the following Σ_E-formula T_E-satisfiable?
  - $f(x) = f(y) \wedge x \neq y$

$x \neq y$ abbreviates $\neg(x = y)$

T_E-Satisfiablility - Example 2

Is the following Σ_E-formula T_E-satisfiable?

$$f(f(f(a))) = a \land f(f(f(f(f(a))))) = a \land f(a) \neq a$$

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$$f(f(f(a))) = a \wedge f(f(f(f(f(a))))) = a \wedge f(a) \neq a$$

1. $f(f(f(f(a)))) = f(a)$ (function congruence)

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1. $f(f(f(f(a)))) = f(a)$ (function congruence)
2. $f(f(f(f(f(a))))) = f(f(a))$ (function congruence)
3. $f(f(a)) = f(f(f(f(f(a)))))$ (symmetry)

T_E-Satisfiablility - Example 2

Is the following Σ_E-formula T_E-satisfiable?

$$f(f(f(a))) = a \wedge f(f(f(f(f(a))))) = a \wedge f(a) \neq a$$

1. $f(f(f(f(a)))) = f(a)$ (function congruence)
2. $f(f(f(f(f(a))))) = f(f(a))$ (function congruence)
3. $f(f(a)) = f(f(f(f(f(a)))))$ (symmetry)
4. $f(f(a)) = a$ (transitivity)

Get Rid of Predicate Congruence

- Transform a Σ_E-formula to a Σ_E-formula without predicates other than $=$
- Example p1
  - $x = y \to (p(x) \leftrightarrow p(y))$ is transformed to
  - $x = y \to ((f_p(x) = \bullet) \leftrightarrow (f_p(y) = \bullet))$
- Example p2
  - $p(x) \land q(x, y) \land q(y, z) \to \neg q(x, z)$ is transformed to
  - $f_p(x) = \bullet \land f_q(x, y) = \bullet \land f_q(y, z) = \bullet \to f_q(x, z) \neq \bullet$

In The Following

- Consider Σ_E-formulae without predicates other than $=$
- T_E-satisfiability of Σ_E-formulae is undecidable
  - Σ_E-formula means quantifier-free Σ_E-formula
- Consider formulae in disjunctive normal form (DNF)

$$(a_1 \wedge a_2 \wedge \ldots \wedge a_n) \vee \ldots \vee (b_1 \wedge b_2 \wedge \ldots \wedge b_m)$$

Congruence Closure Algorithm

Observation

- Applying (symmetry), (reflexivity), (transitivity), and (congruence) to positive literals $s = t$ of a Σ_E-formula F produces more equalities over terms occurring in F
- There are only a finite number of terms in F
- Only a finite number of equalities among these terms are possible
- Then, either
  - some equality is formed that directly contradicts a negative literal $s' \neq t'$ of F; or
  - the propagation of equalities ends without finding a contradiction

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form the congruence closure of $=$

Class

- Consider an equivalence relation R over a set S
- The equivalence class of $s \in S$ under R is the set

$$[s]_R \stackrel{\text{def}}{=} \{s' \in S : sRs'\}$$

- If R is a congruence relation over S, then $[s]_R$ is the congruence class of s

Example of Class

- Consider the set $\mathbb{Z}$ of integers and the equivalence relation $\equiv_2$ such that
  - $m \equiv_2 n \;\text{ iff }\; (m \; mod \; 2) = (n \; mod \; 2)$

$$
\begin{aligned}
[3]_{\equiv_2} &= \{n \in \mathbb{Z} : (n \; mod \; 2) = (3 \; mod \; 2)\} \\
&= \{n \in \mathbb{Z} : (n \; mod \; 2) = 1\} \\
&= \{n \in \mathbb{Z} : n \text{ is odd}\}
\end{aligned}
$$

Partition

A partition P of S is a set of subsets of S that is total,

$$(\cup_{S' \in P} S') = S,$$

and disjoint,

$$\forall S_1, S_2 \in P.\ S_1 \neq S_2 \rightarrow S_1 \cap S_2 = \varnothing$$

Quotient

- The quotient S/R of S by the equivalence (congruence) relation R is a partition of S: it is a set of equivalence (congruence) classes
- $S/R = \{[s]_R : s \in S\}$

Example of Quotient

- The quotient $\mathbb{Z}/\equiv_2$ is a partition: it is the set of equivalence classes
  - $\{\{n \in \mathbb{Z} : n \text{ is odd}\}, \{n \in \mathbb{Z} : n \text{ is even}\}\}$

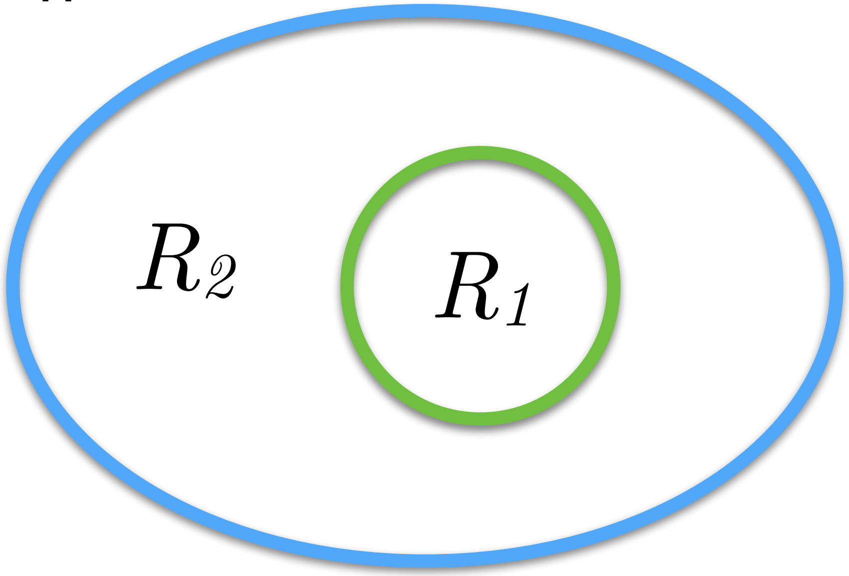
Equivalence Relation, Partition, and Quotient

- An equivalence relation R induces a partition S/R of S
- A given partition P of S induces an equivalence relation over S
  - $s_1 R s_2$ iff for some S' ∈ P, both s_1, s_2 ∈ S'

Relation Refinement

- Consider two binary relations R_1 and R_2 over the set S

- R_1 is a refinement of R_2, or $R_1 < R_2$, if
  - $\forall s_1, s_2 \in S.\ s_1 R_1 s_2 \rightarrow s_1 R_2 s_2$



- We also say that R_1 refines R_2

- Viewing the relations as sets of pairs, $R_1 \subseteq R_2$

Example 1 of Relation Refinement

- $S = \{a,\ b\}$
- $R_1 : \{aR_1b\}$
- $R_2 : \{aR_2b,\ bR_2b\}$
- $R_1 < R_2$

Example 2 of Relation Refinement

- Consider set S
- $R_1 : \{sR_1s : s \in S\}$
- $R_2 : \{sR_2t : s,t \in S \}$
- $R_1 \prec R_2$

Example 2 of Relation Refinement

- Consider set S
- $R_1 : \{sR_1s : s \in S\}$ $P_1 : \{\{s\} : s \in S\}$
- $R_2 : \{sR_2t : s,t \in S \}$
- $R_1 \prec R_2$

Example 2 of Relation Refinement

- Consider set S
- $R_1 : \{sR_1s : s \in S\}$ $\qquad$ $P_1 : \{\{s\} : s \in S\}$
- $R_2 : \{sR_2t : s,t \in S\}$ $\qquad$ $P_2 : \{S\}$
- $R_1 < R_2$

Example 3 of Relation Refinement

- Consider the set $\mathbb{Z}$
- $R_1 : \{xR_1y : x\ mod\ 2 = y\ mod\ 2\}$
- $R_2 : \{xR_1y : x\ mod\ 4 = y\ mod\ 4\}$
- $R_2 < R_1$

Closure

- The equivalence closure R^E of the binary relation R over S is the equivalence relation such that
  - R refines R^E: $R < R^E$;
  - for all other equivalence relations R' such that $R < R$', either
    - $R' = R^E$, or
    - $R^E < R'$
- R^E is the smallest equivalence relation that covers R

Example of Equivalence Closure

- Then,
  - $aRb,\ bRc,\ dRd \in R^E$ (since $R \subseteq R^E$);
  - $aRa,\ bRb,\ cRc \in R^E$ (by reflexivity);
  - $bRa,\ cRb \in R^E$ (by symmetry);
  - $aRc \in R^E$ (by transitivity);
  - $cRa \in R^E$ (by symmetry);

$$S = \{a,\ b,\ c,\ d\}$$

$$R = \{aRb,\ bRc,\ dRd\}$$

- Hence
  - $R^E = \{aRb,\ bRa,\ aRa,\ bRb,\ bRc,\ cRb,\ cRc,\ aRc,\ cRa,\ dRd\}$

Congruence Closure

- The congruence closure R^C of R is the smallest congruence relation that covers R

Congruence Closure

- The congruence closure R^C of R is the smallest congruence relation that covers R

Compute the congruence closure of a term set

Subterm Set

- Subterm set S_F of Σ_E-formula F is the set that contains precisely the subterms of F

- Example:

  - $F : f(a,\ b) = a \wedge f(f(a,\ b),\ b) \neq a$

  - $S_F = \{a,\ b,\ f(a,\ b),\ f(f(a,\ b),\ b)\}$

Congruence Relation over Subterm Set

$$F : s_1 = t_1 \wedge \ldots \wedge s_m = t_m \wedge s_{m+1} \neq t_{m+1} \wedge \ldots \wedge s_n \neq t_n$$

- F is T_E-satisfiable iff there exists a congruence relation $\sim$ over S_F such that
    - for each $i \in \{1, \ldots, m\}$, $s_i \sim t_i$;
    - for each $i \in \{m + 1, \ldots, n\}$, $s_i \nsim t_i$

T_E-interpretation

- The congruence relation $\sim$ defines a T_E-interpretation I : $(D_{\mathrm{I}},\ a_I)$ of F
    - D_I consists of $|S_F\ /\ \sim|$ elements
    - a_I assigns elements of D_I to the terms of S_F in a way that respects $\sim$
    - a_I assigns to $=$ a binary relation over D_I that behaves like $\sim$
- We abbreviate $(D_I,\ a_I) \vDash F$ with $\sim\ \vDash\ F$

Congruence Closure Algorithm

$$F : s_1 = t_1 \wedge \ldots \wedge s_m = t_m \wedge s_{m+1} \neq t_{m+1} \wedge \ldots \wedge s_n \neq t_n$$

1. Construct the congruence closure ~ of

   $$\{s_1 = t_1, \ldots, s_m = t_m\}$$

   over the subterm set S_F

2. If $s_i \sim t_i$ for any $i \in \{m + 1, \ldots, n\}$, return unsatisfiable

3. Otherwise, $\sim \models F$, so return satisfiable

Step 1

- Begin with $\sim_0$ given by the partition $\{\{s\} : s \in S_F\}$
- Import $s_i = t_i$ by merging the congruence classes $[s_i]_{\sim i\text{-}1}$ and $[t_i]_{\sim i\text{-}1}$
  - Form the union of $[s_i]_{\sim i\text{-}1}$ and $[t_i]_{\sim i\text{-}1}$
  - Propagate new congruences that arise within the union

Example 1 of Congruence Closure Algorithm

$$F : f(a,\ b) = a \wedge f(f(a,\ b),\ b) \neq a$$

- $\{\{a\},\ \{b\},\ \{f(a,\ b)\},\ \{f(f(a,\ b),\ b)\}\}$

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(function congruence)

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- T_E-unsatisfiable

Example 2 of Congruence Closure Algorithm

$$F: f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$$

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T_E-unsatisfiable

Example 3 of Congruence Closure Algorithm

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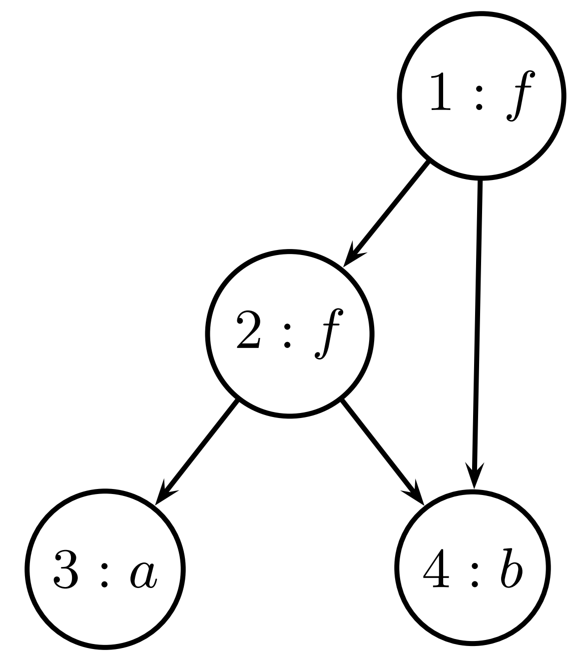
- $\{\{x\}, \{y\}, \{f(x), f(y)\}\}$
- T_E-satisfiable

Implementation

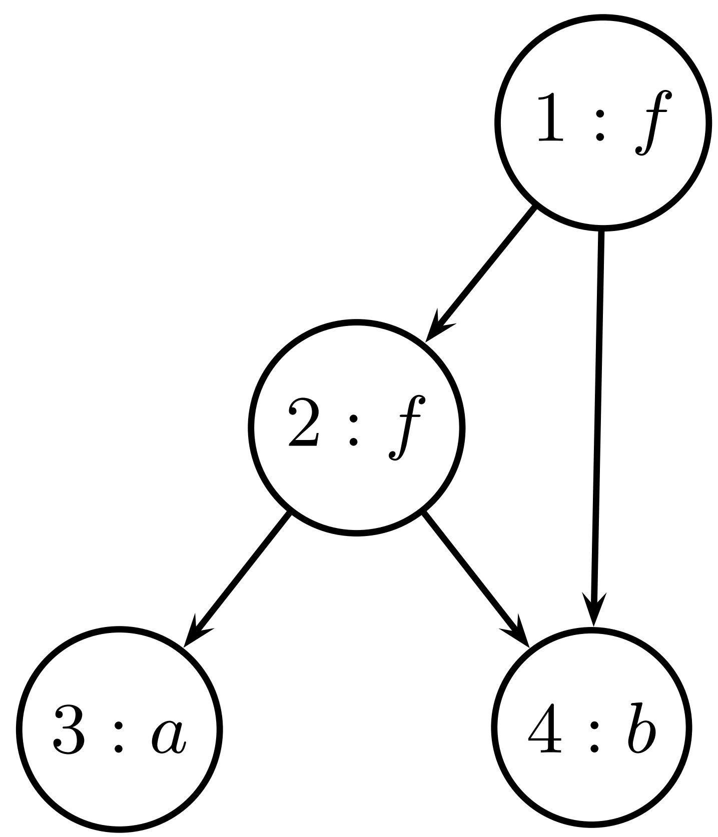
DAG

- A directed graph $G : \langle N,\ E\rangle$
  - nodes $N = \{n_1,\ n_2,\ ...,\ n_k\}$
  - edges $E = \{...,\ \langle n_i,\ n_j\rangle,\ ...\}$



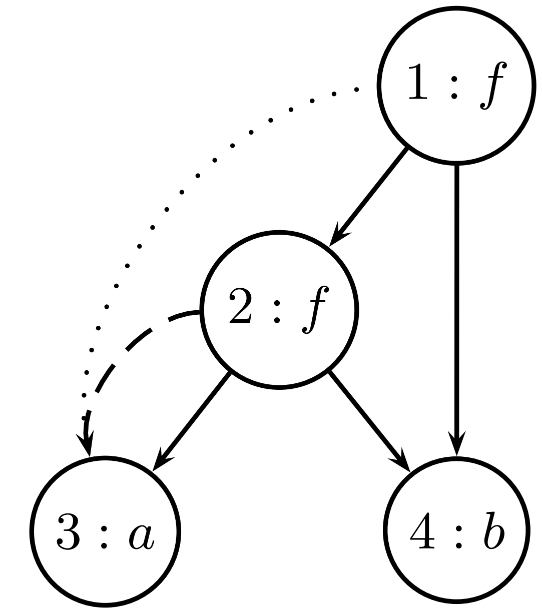
- A directed acyclic graph (DAG) is a directed graph containing no loop (or cycle)

Subterm Set as DAG



$$\{a,\ b,\ f(a,\ b),\ f(f(a,\ b),\ b)\}$$

Node



type node = {

id : id (unique identification number)

fn : string (constant or function symbol)

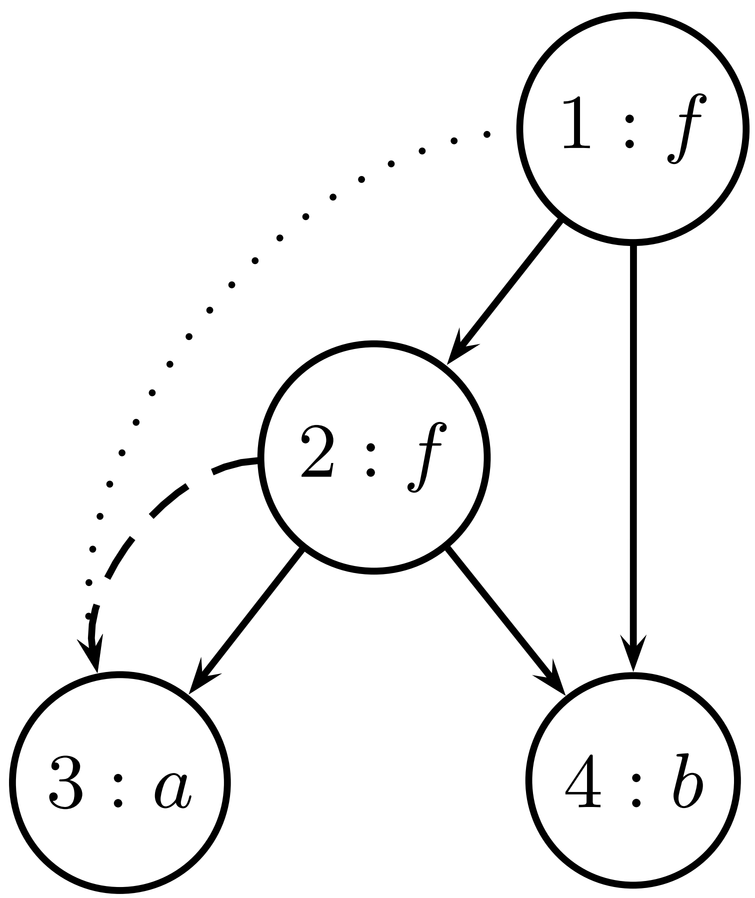
args : id list (identification numbers of the function arguments)

mutable find : id (another node in its congruence class)
(following a chain of find references leads to the representative)

mutable ccpar : id set (congruence closure parents, ∅ for non-representative nodes)

}

DAG as Partition

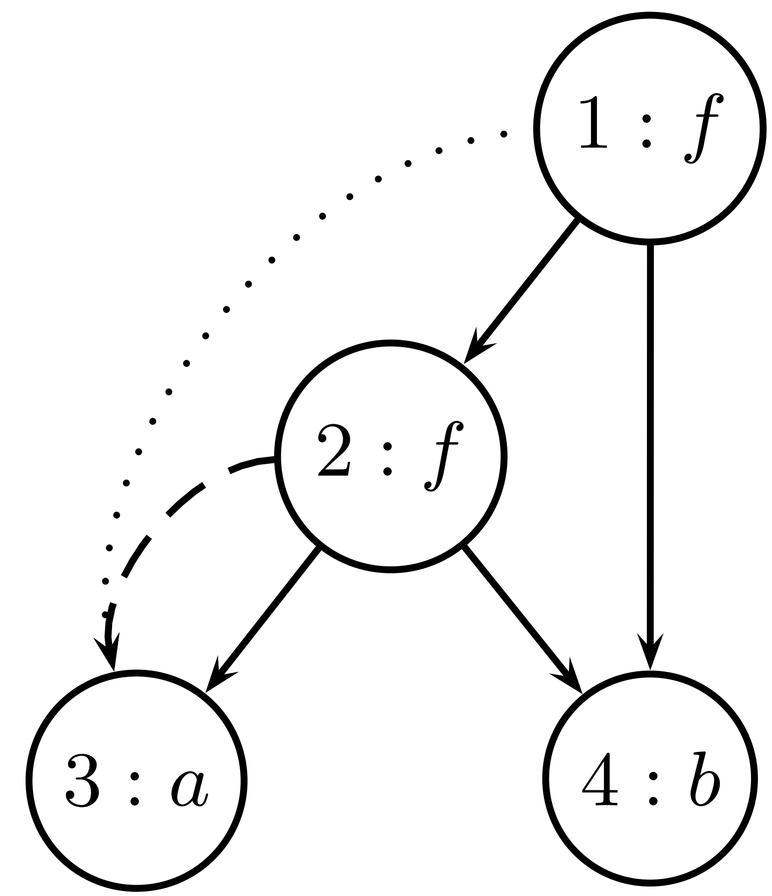


$$
\begin{array}{l}
\text{node } 2 = \{ \\
\quad \text{id} = 2; \\
\quad \text{fn} = f; \\
\quad \text{args} = [3;\ 4]; \\
\quad \text{find} = 3; \\
\quad \text{ccpar} = \varnothing; \\
\}
\end{array}
\qquad
\begin{array}{l}
\text{node } 3 = \{ \\
\quad \text{id} = 3; \\
\quad \text{fn} = a; \\
\quad \text{args} = []; \\
\quad \text{find} = 3; \\
\quad \text{ccpar} = \{1,\ 2\}; \\
\}
\end{array}
$$

Partition: $\{\{f(f(a,\ b),\ b),\ f(a,\ b),\ a\},\ \{b\}\}$

Union-Find Algorithm - NODE

NODE i returns the node n with id i



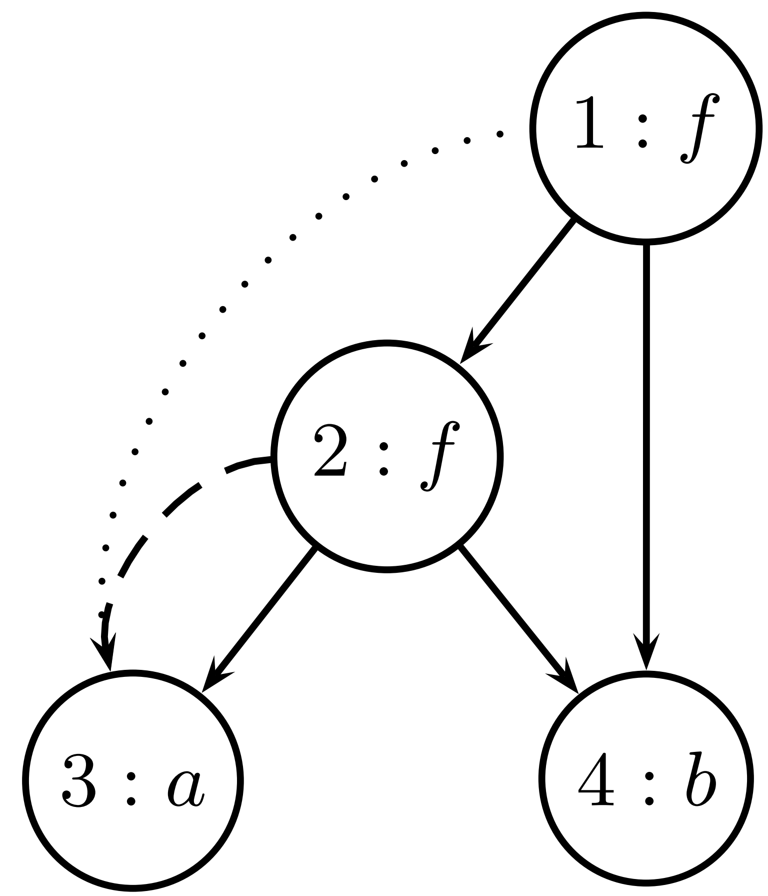
$$(\text{NODE } i).\text{id} = i$$
$$(\text{NODE } 2).\text{find} = 3$$

Union-Find Algorithm - FIND

$$\text{let rec FIND } i =$$

$$\quad \text{let } n = \text{NODE } i \text{ in}$$

$$\quad \text{if } n.\text{find} = i \text{ then } i \text{ else FIND } n.\text{find}$$



$$\text{FIND } 2 = 3$$

$$\text{FIND } 1 = 3$$

Union-Find Algorithm - UNION

let UNION i_1 i_2 =

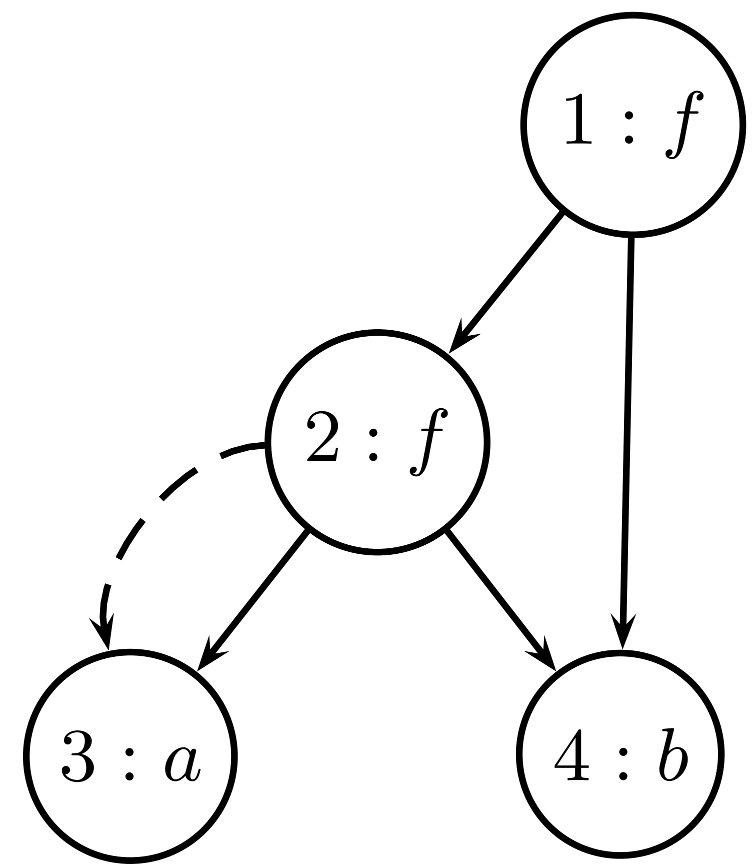
let n_1 = NODE (FIND i_1) in

let n_2 = NODE (FIND i_2) in

n_1.find $\leftarrow$ n_2.find;

n_2.ccpar $\leftarrow$ n_1.ccpar $\cup$ n_2.ccpar;

n_1.ccpar $\leftarrow \varnothing$



Union-Find Algorithm - UNION

let UNION i_1 i_2 =

let n_1 = NODE (FIND i_1) in

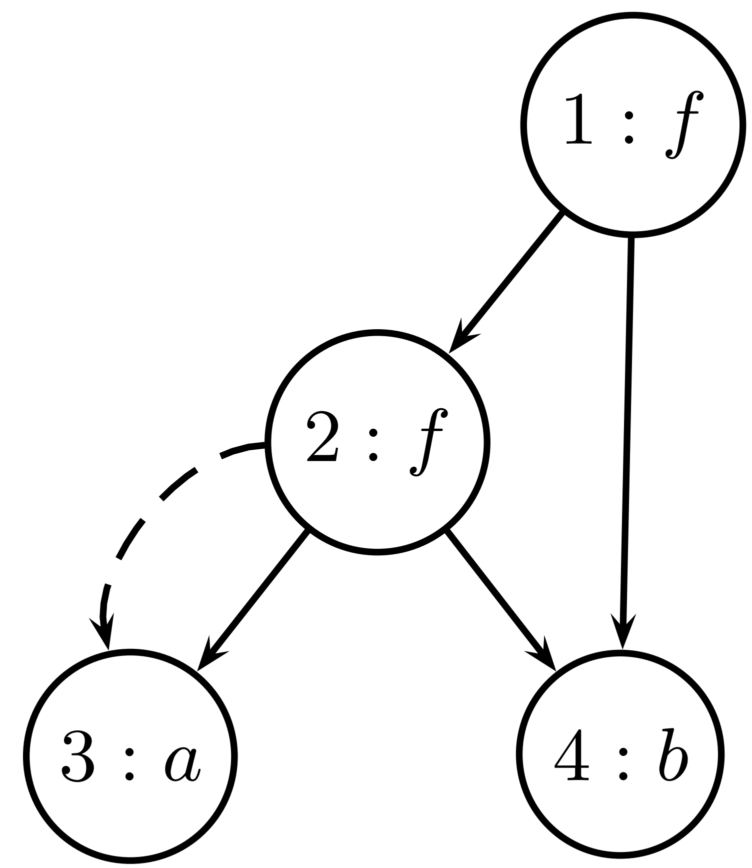
let n_2 = NODE (FIND i_2) in

n_1.find $\leftarrow$ n_2.find;

n_2.ccpar $\leftarrow$ n_1.ccpar $\cup$ n_2.ccpar;

n_1.ccpar $\leftarrow$ $\varnothing$

UNION 1 2



Union-Find Algorithm - UNION

let UNION i_1 i_2 =

let n_1 = NODE (FIND i_1) in

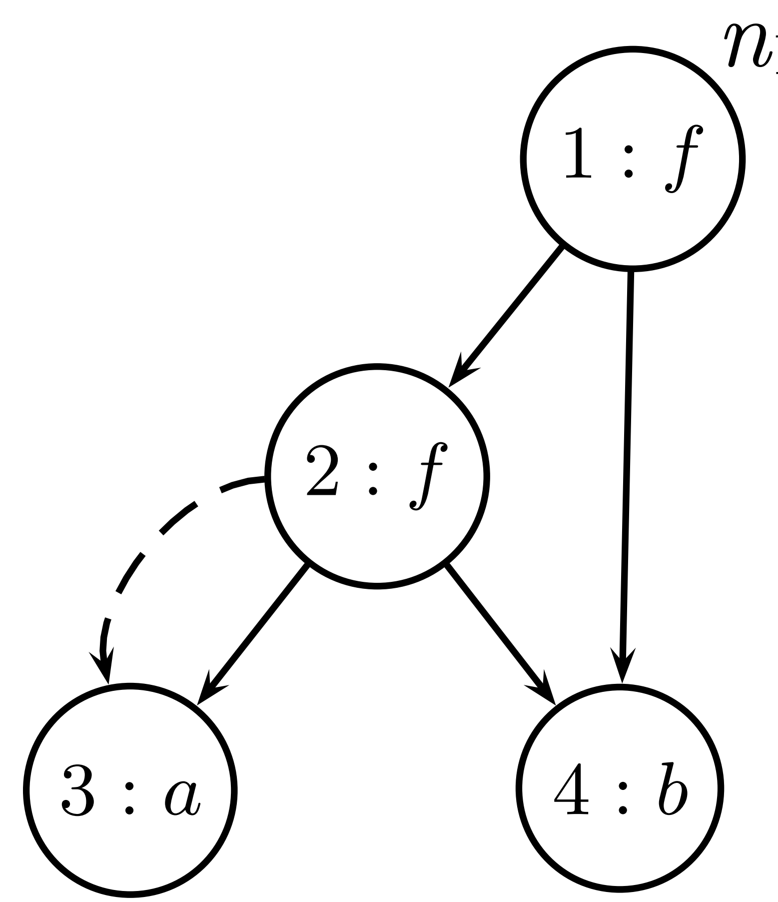
let n_2 = NODE (FIND i_2) in

n_1.find $\leftarrow$ n_2.find;

n_2.ccpar $\leftarrow$ n_1.ccpar $\cup$ n_2.ccpar;

n_1.ccpar $\leftarrow \varnothing$

UNION 1 2



Union-Find Algorithm - UNION

let UNION i_1 i_2 =

let n_1 = NODE (FIND i_1) in

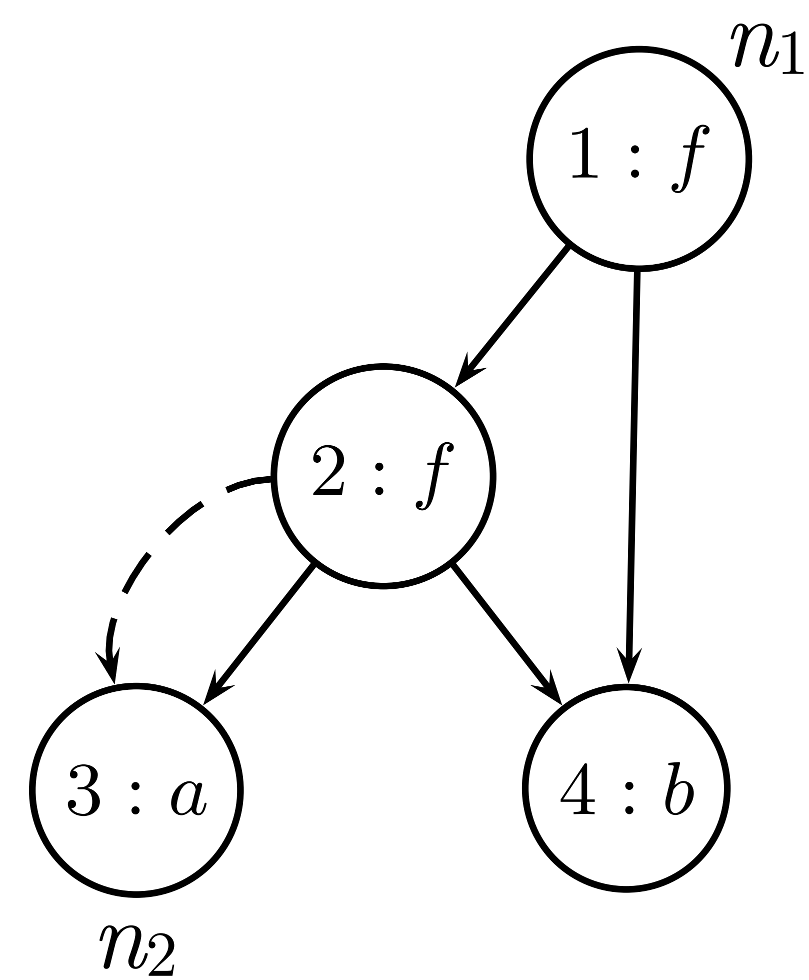
let n_2 = NODE (FIND i_2) in

n_1.find $\leftarrow$ n_2.find;

n_2.ccpar $\leftarrow$ n_1.ccpar $\cup$ n_2.ccpar;

n_1.ccpar $\leftarrow$ $\varnothing$

UNION 1 2



Union-Find Algorithm - UNION

let UNION i_1 i_2 =

let n_1 = NODE (FIND i_1) in

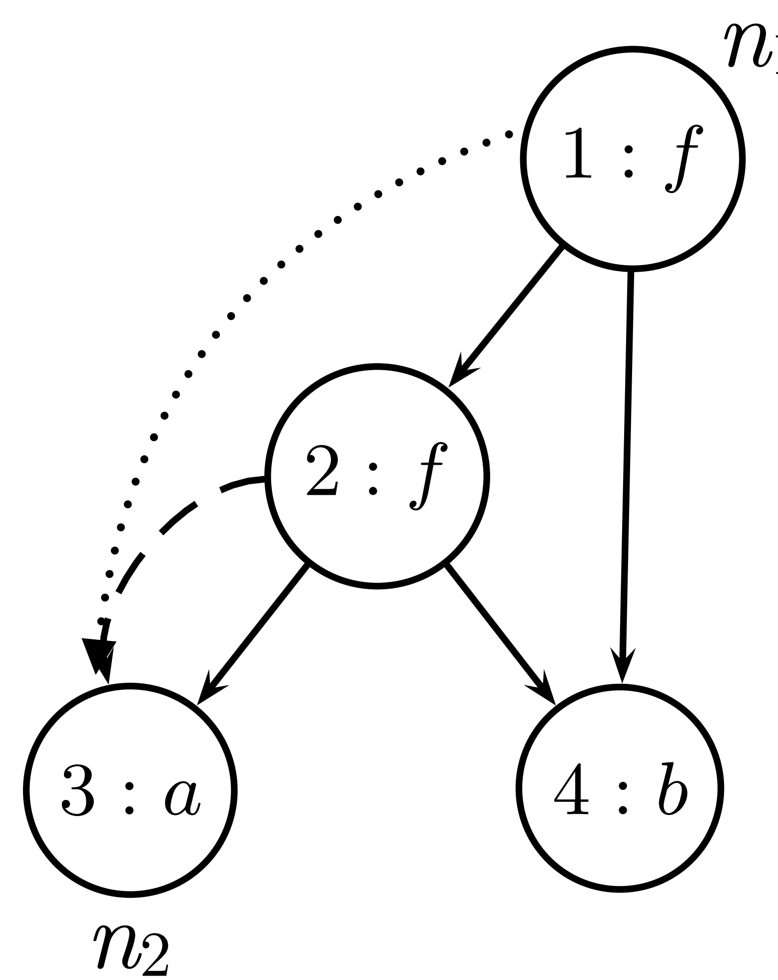
let n_2 = NODE (FIND i_2) in

n_1.find $\leftarrow$ n_2.find;

n_2.ccpar $\leftarrow$ n_1.ccpar $\cup$ n_2.ccpar;

n_1.ccpar $\leftarrow \varnothing$

UNION 1 2



Union-Find Algorithm - CCPAR

let $\mathrm{CCPAR}\ i =$

$\quad (\mathrm{NODE}\ (\mathrm{FIND}\ i)).\mathrm{ccpar}$

Congruence Closure Algorithm - CONGRUENT

let CONGRUENT i_1 i_2 =

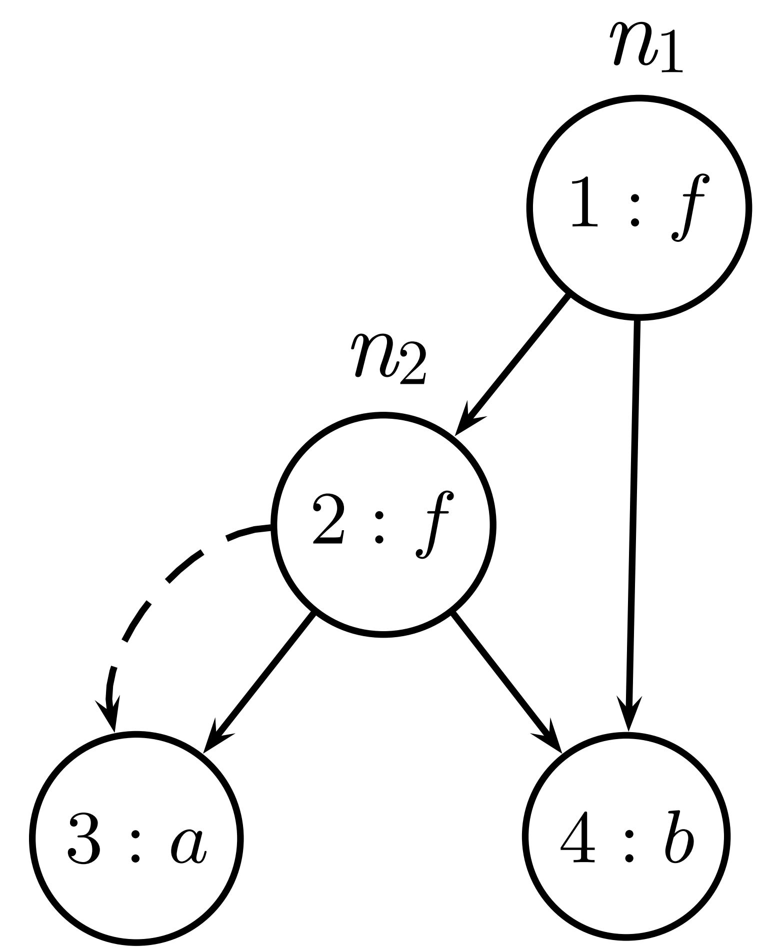
let n_1 = NODE i_1 in

let n_2 = NODE i_2 in

$n_1.\mathrm{fn} = n_2.\mathrm{fn}$

$\wedge\ |n_1.\mathrm{args}| = |n_2.\mathrm{args}|$

$\wedge\ \forall i \in \{1, ..., |n_1.\mathrm{args}|\}.\ \mathrm{FIND}\ n_1.\mathrm{args}[i] = \mathrm{FIND}\ n_2.\mathrm{args}[i]$



Congruence Closure Algorithm - MERGE

let rec MERGE i_1 i_2 =

if FIND $i_1 \neq$ FIND i_2 then begin

let P_1 = CCPAR i_1 in

let P_2 = CCPAR i_2 in

UNION i_1 i_2;

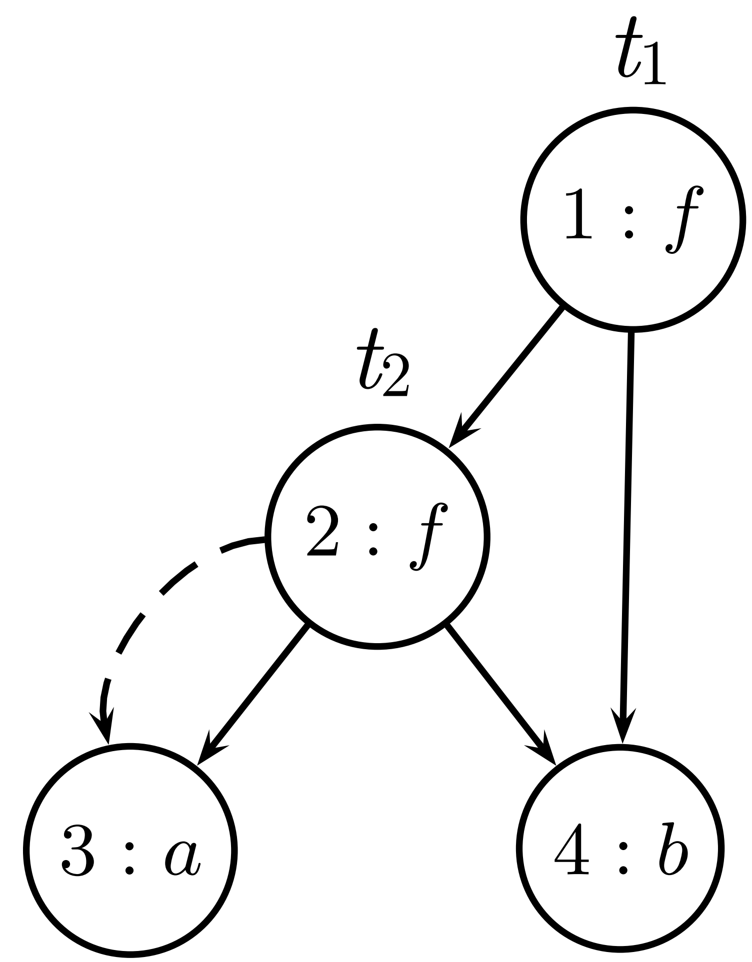
foreach $t_1, t_2 \in P_1 \times P_2$ do

if FIND $t_1 \neq$ FIND $t_2 \wedge$ CONGRUENT t_1 t_2

then MERGE t_1 t_2

done

end



Decision Procedure for T_E-Satisfiability

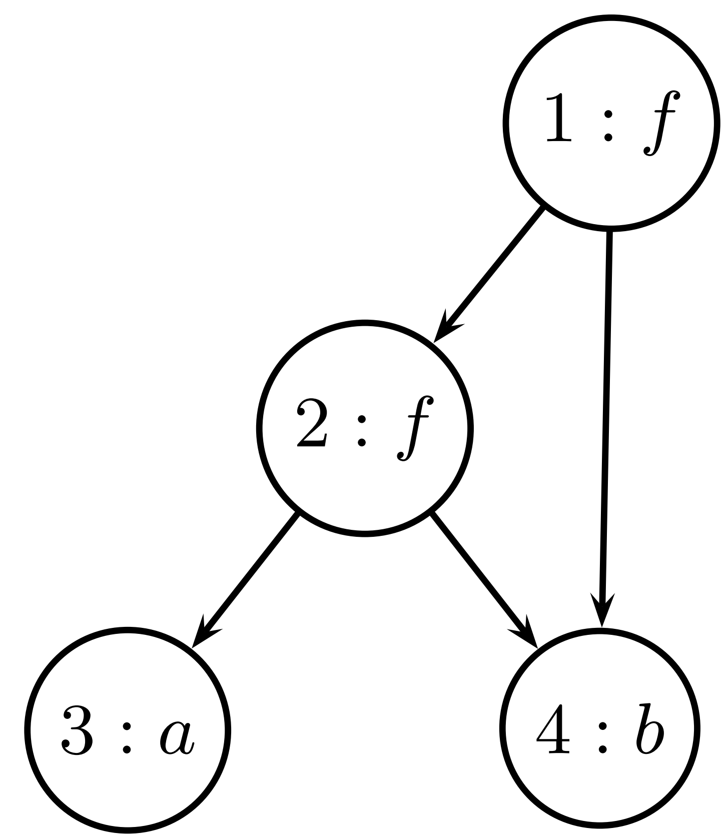
$$F: s_1 = t_1 \wedge \ldots \wedge s_m = t_m \wedge s_{m+1} \neq t_{m+1} \wedge \ldots \wedge s_n \neq t_n$$

1. Construct the initial DAG for the subterm set S_F
2. For $i \in \{1, \ldots, m\}$, $\mathrm{MERGE}\ s_i\ t_i$
3. If $\mathrm{FIND}\ s_i = \mathrm{FIND}\ t_i$ for some i $\in \{m+1, \ldots, n\}$, return unsatisfiable
4. Otherwise, return satisfiable

Deciding T_E-Satisfiability Example 1

$$F : f(a,\ b) = a \land f(f(a,\ b),\ b) \neq a$$

$$S_F = \{a,\ b,\ f(a,\ b),\ f(f(a,\ b),\ b)\}$$

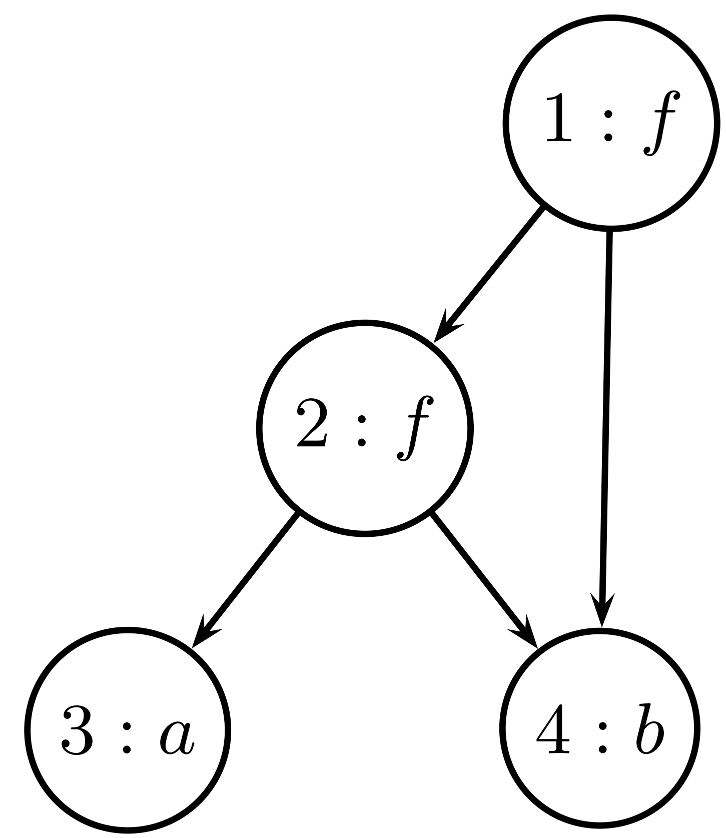


Deciding T_E-Satisfiability Example 1

$$F : f(a,\ b) = a \wedge f(f(a,\ b),\ b) \neq a$$

$$S_F = \{a,\ b,\ f(a,\ b),\ f(f(a,\ b),\ b)\}$$

1.MERGE 2 3



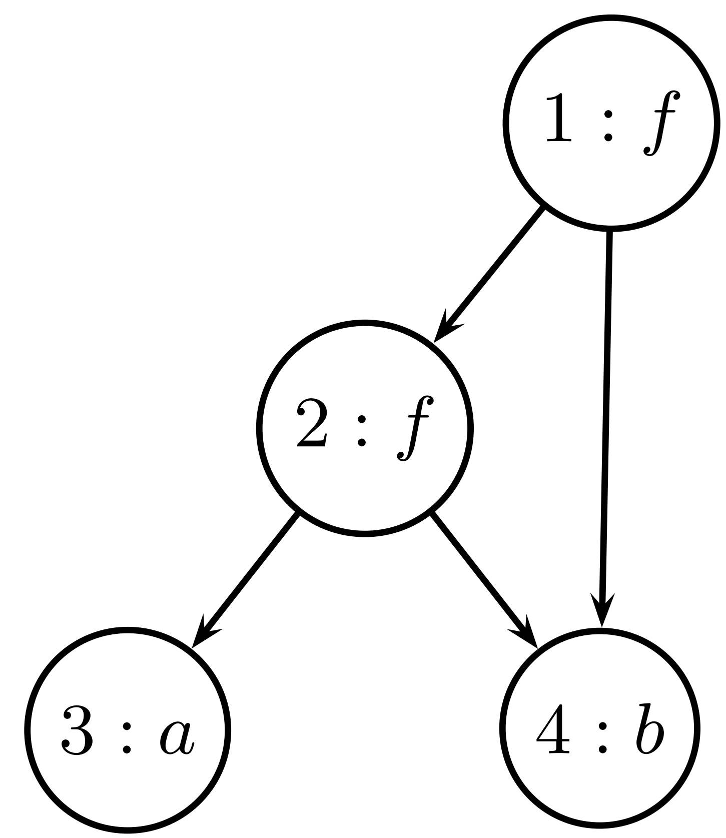
Deciding T_E-Satisfiability Example 1

$$F : f(a,\ b) = a \wedge f(f(a,\ b),\ b) \neq a$$

$$S_F = \{a,\ b,\ f(a,\ b),\ f(f(a,\ b),\ b)\}$$

$1.\mathrm{MERGE}\ 2\ 3$

$(1) P_2 = \mathrm{CCPAR}\ 2 = \{1\}$



Deciding T_E-Satisfiability Example 1

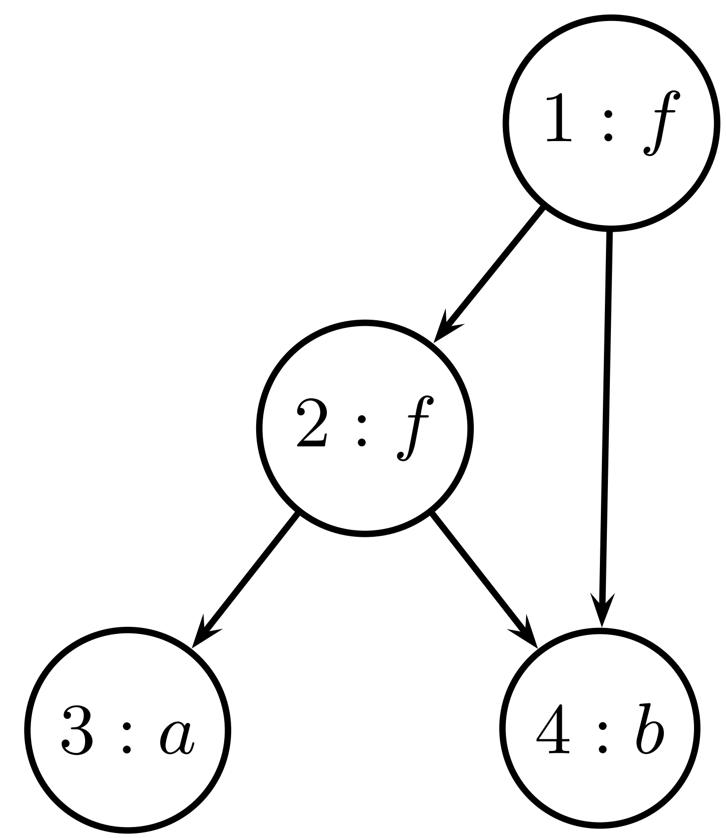
$$F : f(a,\ b) = a \wedge f(f(a,\ b),\ b) \neq a$$

$$S_F = \{a,\ b,\ f(a,\ b),\ f(f(a,\ b),\ b)\}$$

1. MERGE 2 3

   (1) $P_2 = \text{CCPAR } 2 = \{1\}$

   (2) $P_3 = \text{CCPAR } 3 = \{2\}$

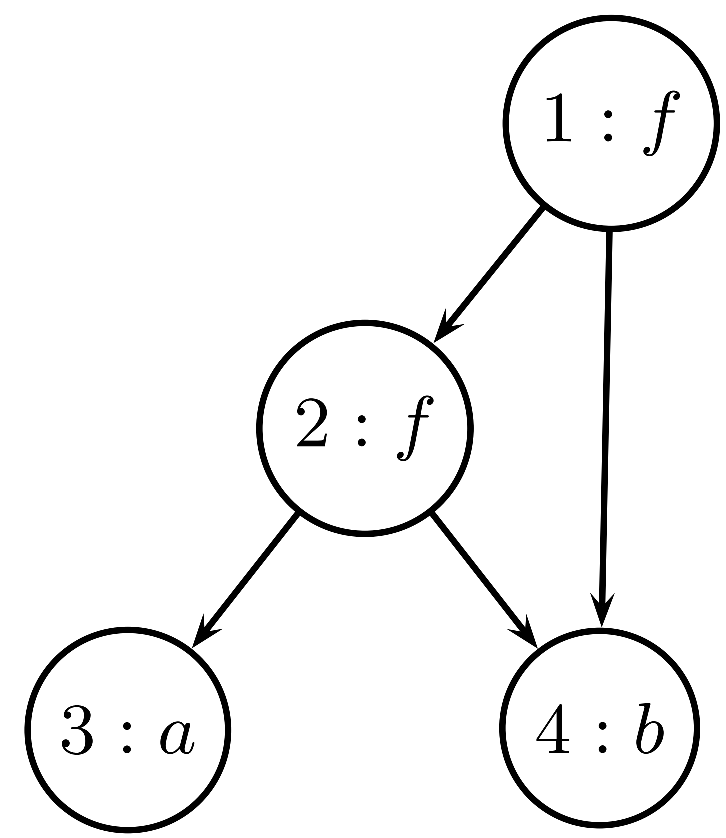


Deciding T_E-Satisfiability Example 1

$$F : f(a,\ b) = a \wedge f(f(a,\ b),\ b) \neq a$$

$$S_F = \{a,\ b,\ f(a,\ b),\ f(f(a,\ b),\ b)\}$$

1. MERGE 2 3
   (1) $P_2 = \text{CCPAR } 2 = \{1\}$
   (2) $P_3 = \text{CCPAR } 3 = \{2\}$
   (3) UNION 2 3



Deciding T_E-Satisfiability Example 1

$$F : f(a,\ b) = a \wedge f(f(a,\ b),\ b) \neq a$$

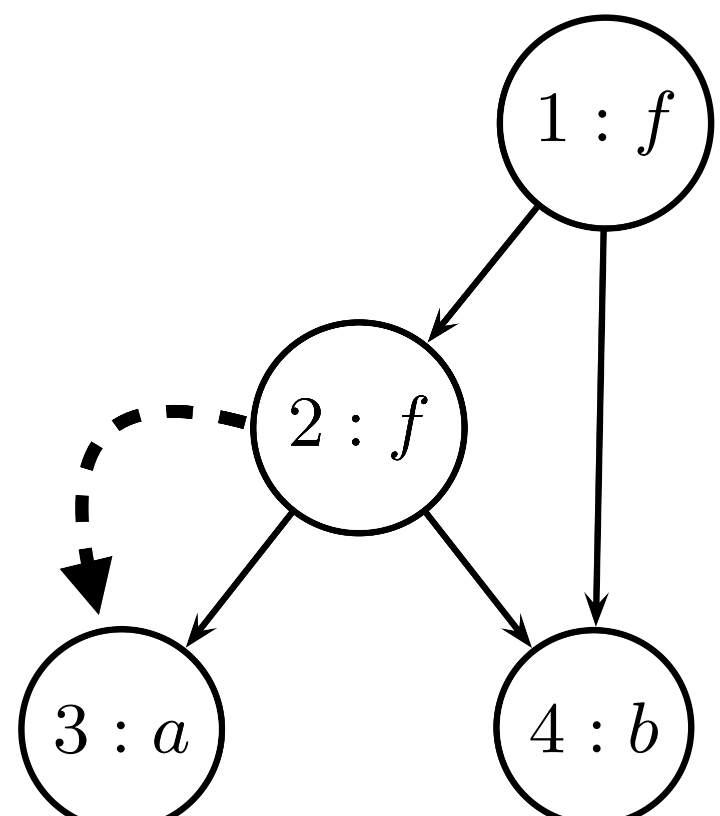
$$S_F = \{a,\ b,\ f(a,\ b),\ f(f(a,\ b),\ b)\}$$

1.MERGE 2 3

(1)$P_2 = \text{CCPAR } 2 = \{1\}$

(2)$P_3 = \text{CCPAR } 3 = \{2\}$

(3)UNION 2 3



Deciding T_E-Satisfiability Example 1

$$F : f(a,\ b) = a \land f(f(a,\ b),\ b) \neq a$$

$$S_F = \{a,\ b,\ f(a,\ b),\ f(f(a,\ b),\ b)\}$$

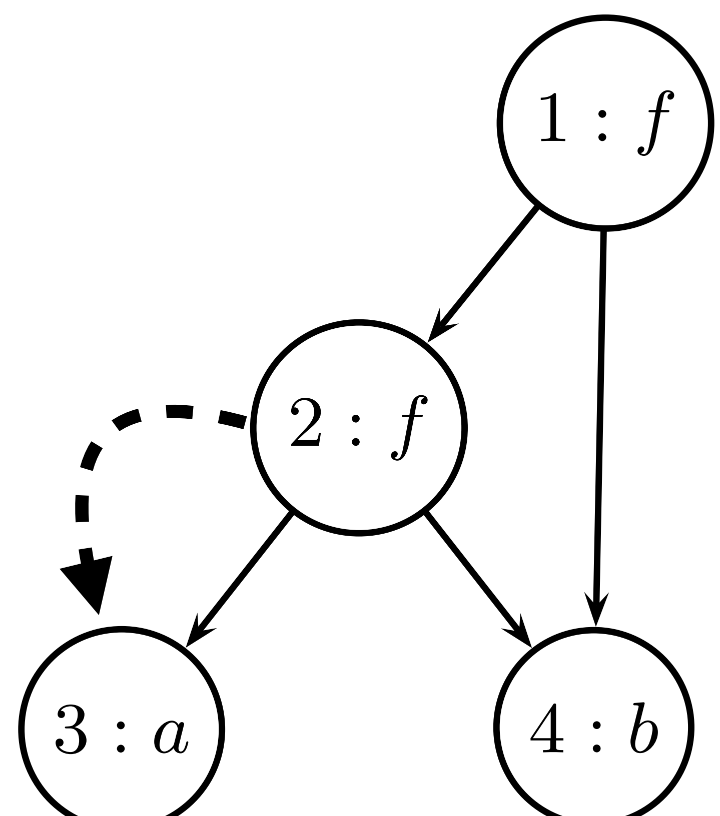
1.MERGE 2 3

(1)$P_2 = \text{CCPAR } 2 = \{1\}$

(2)$P_3 = \text{CCPAR } 3 = \{2\}$

(3)UNION 2 3

(4)MERGE 1 2



Deciding T_E-Satisfiability Example 1

$$F : f(a,\ b) = a \wedge f(f(a,\ b),\ b) \neq a$$

$$S_F = \{a,\ b,\ f(a,\ b),\ f(f(a,\ b),\ b)\}$$

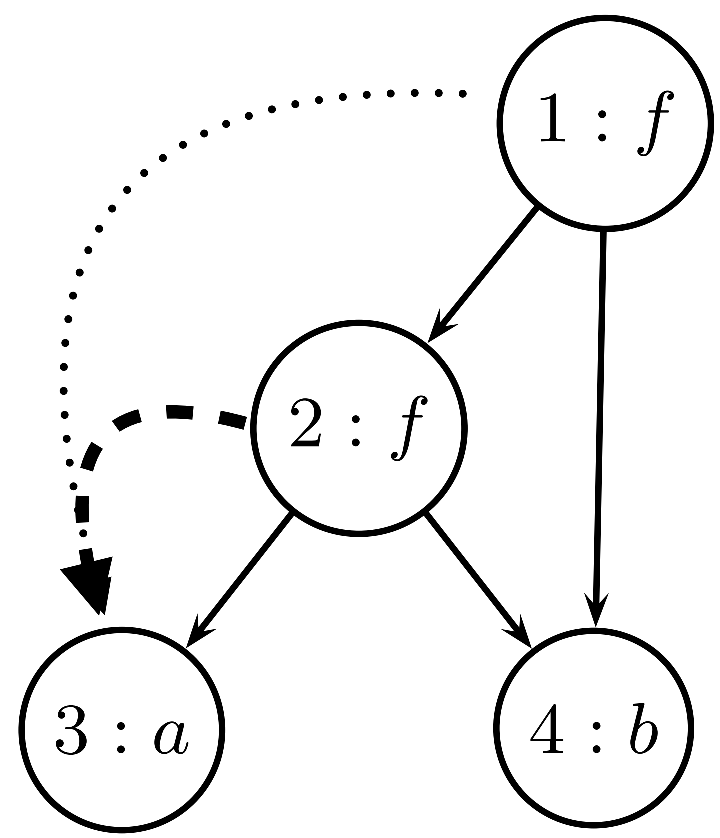
1.MERGE 2 3

(1)$P_2 = \text{CCPAR } 2 = \{1\}$

(2)$P_3 = \text{CCPAR } 3 = \{2\}$

(3)UNION 2 3

(4)MERGE 1 2



Deciding T_E-Satisfiability Example 1

$$F : f(a,\ b) = a \wedge f(f(a,\ b),\ b) \neq a$$

$$S_F = \{a,\ b,\ f(a,\ b),\ f(f(a,\ b),\ b)\}$$

1.MERGE 2 3

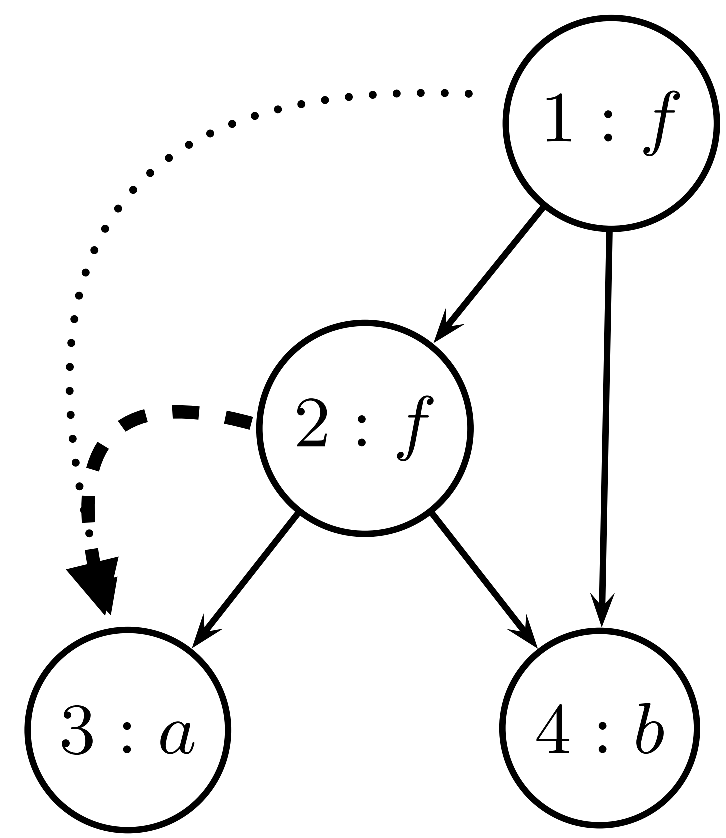
(1) $P_2 = \text{CCPAR } 2 = \{1\}$

(2) $P_3 = \text{CCPAR } 3 = \{2\}$

(3) UNION 2 3

(4) MERGE 1 2

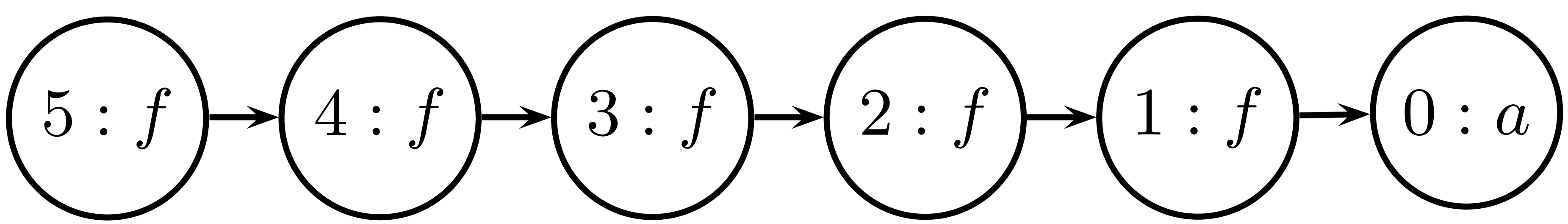
T_E-unsatisfiable



Deciding T$_E$-Satisfiability Example 2

$$F : f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$$

$$S_F = \{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a))\}$$

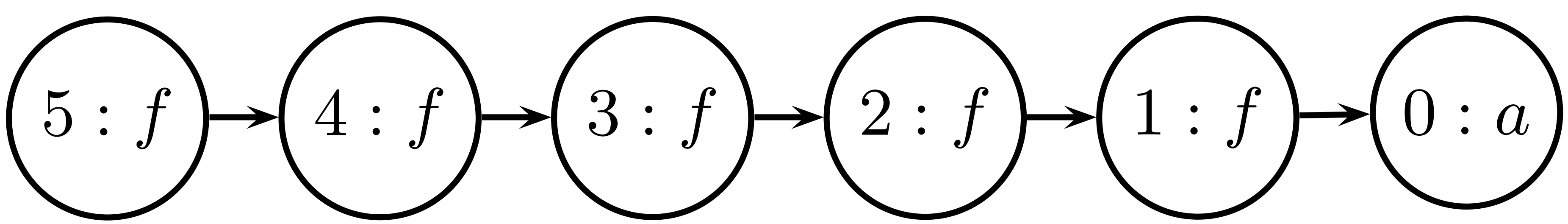


Deciding T_E-Satisfiability Example 2

$$F : f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$$

$$S_F = \{a,\ f(a),\ f^2(a),\ f^3(a),\ f^4(a),\ f^5(a))\}$$

1. MERGE 3 0

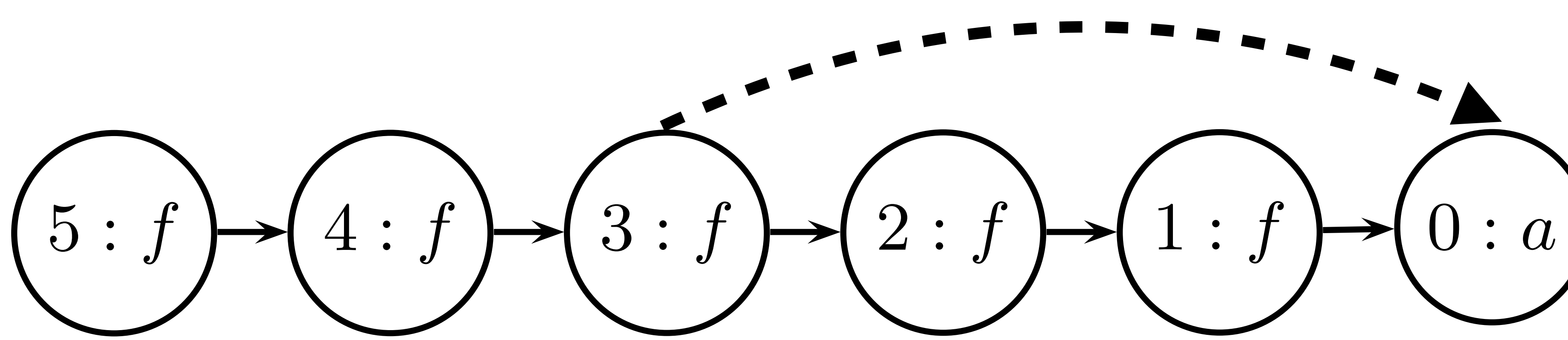


Deciding T_E-Satisfiability Example 2

$$F : f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$$

$$S_F = \{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a))\}$$

1. MERGE 3 0

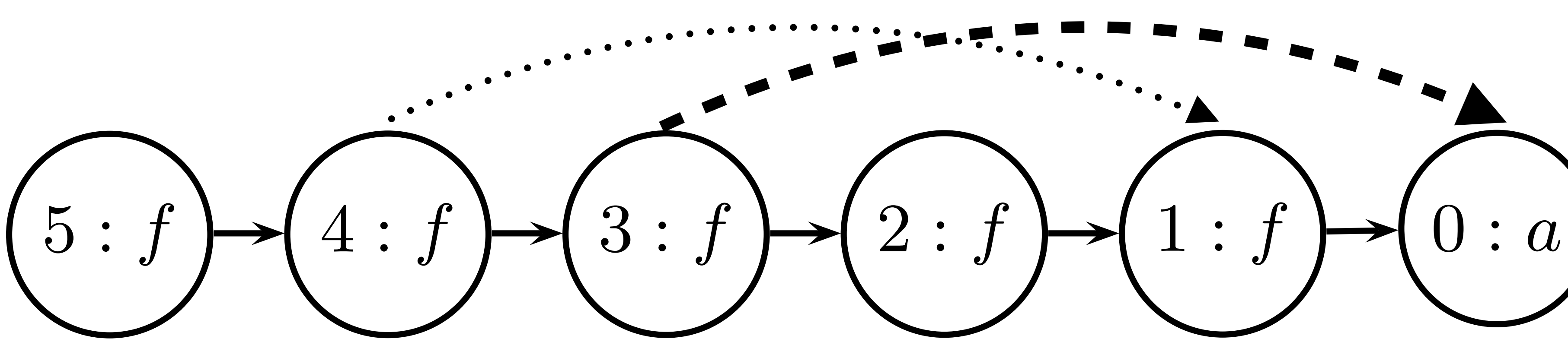


Deciding T_E-Satisfiability Example 2

$$F : f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$$

$$S_F = \{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a))\}$$

1. MERGE 3 0

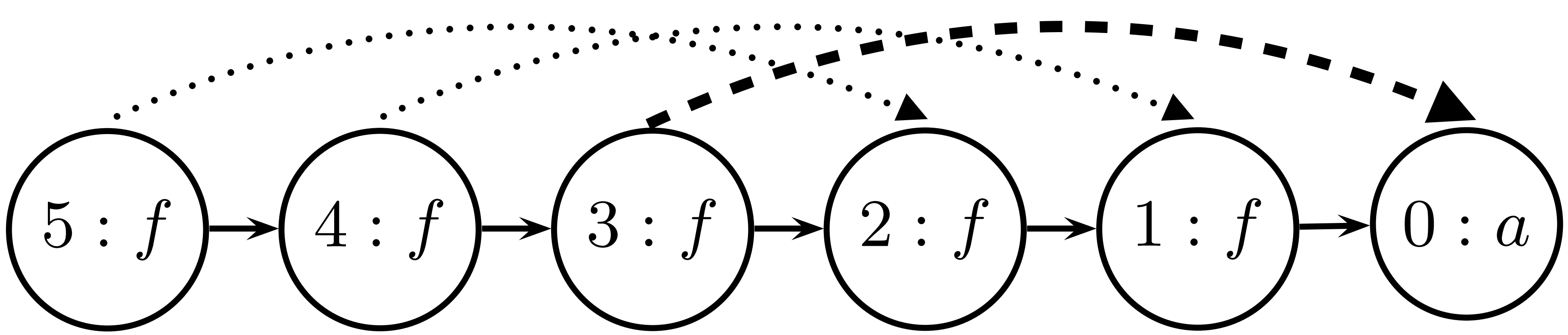


Deciding T_E-Satisfiability Example 2

$$F : f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$$

$$S_F = \{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a))\}$$

1. MERGE 3 0

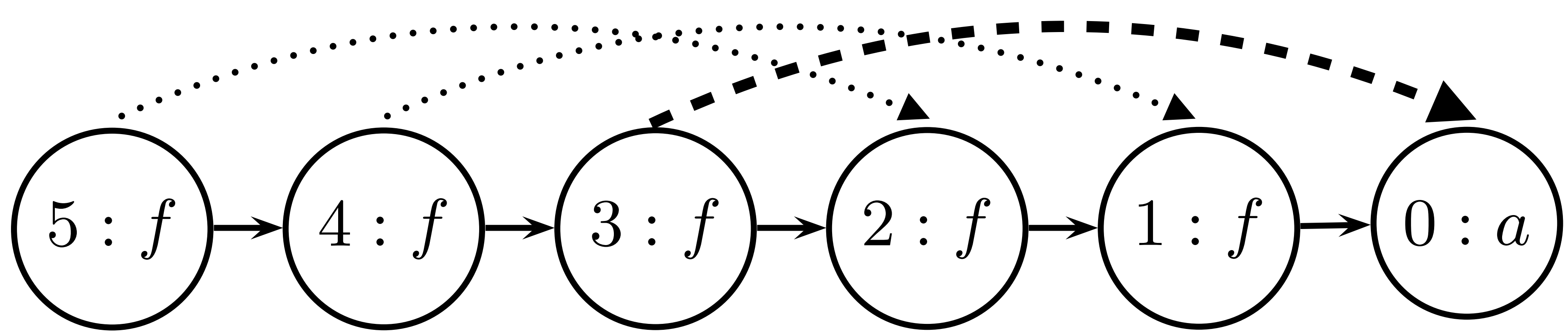


Deciding T_E-Satisfiability Example 2

$$F : f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$$

$$S_F = \{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a))\}$$

1. MERGE 3 0
2. MERGE 5 0

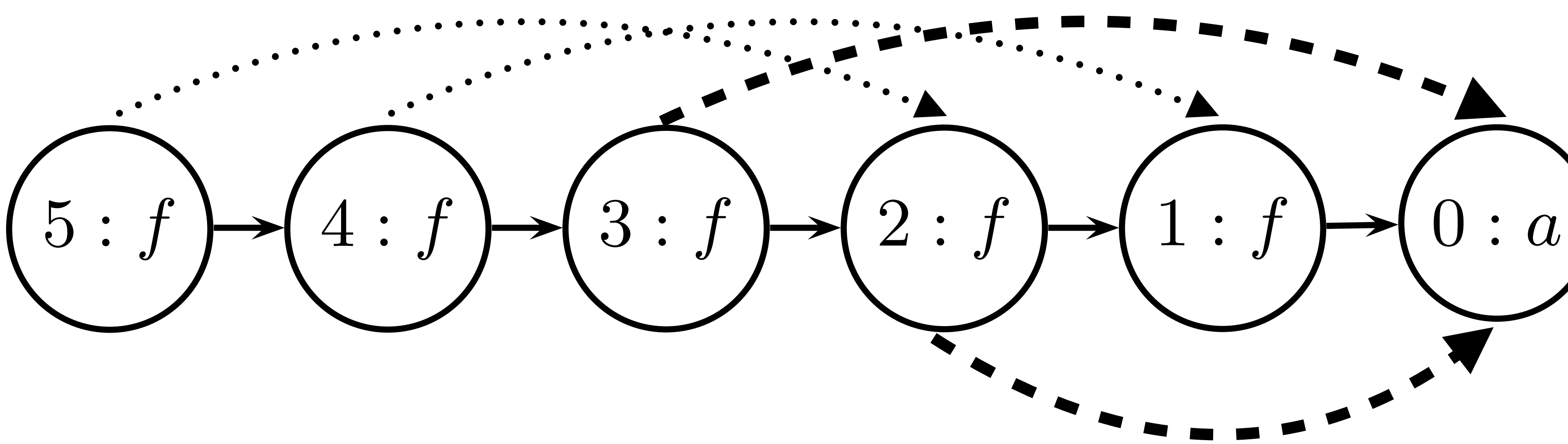


Deciding T$_E$-Satisfiability Example 2

$$F : f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$$

$$S_F = \{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a))\}$$

1. MERGE 3 0
2. MERGE 5 0

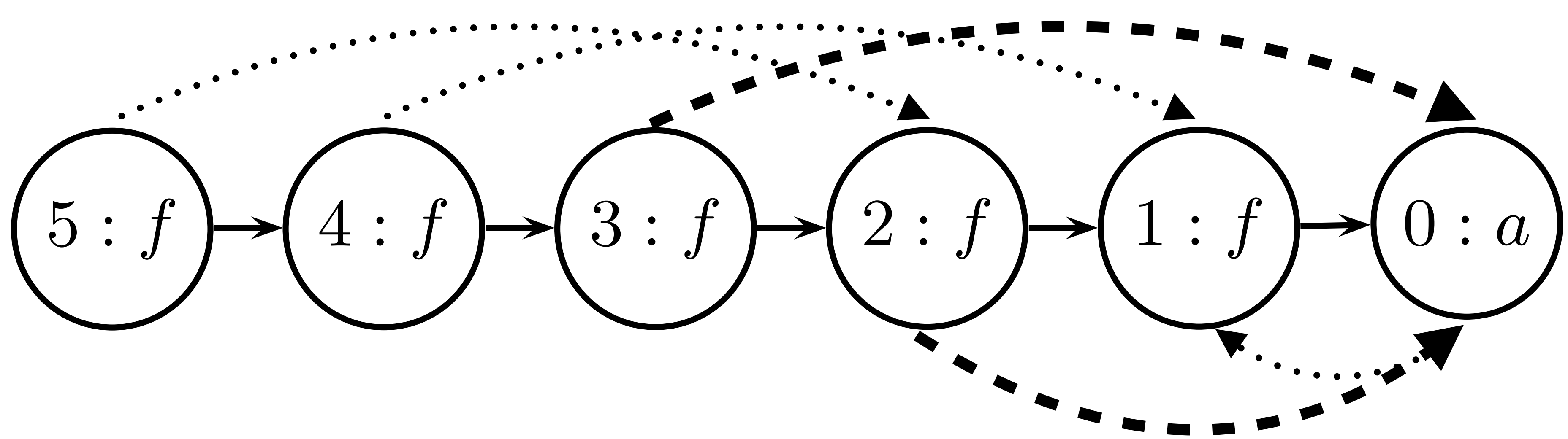


Deciding T_E-Satisfiability Example 2

$$F : f^3(a) = a \land f^5(a) = a \land f(a) \neq a$$

$$S_F = \{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a))\}$$

1. MERGE 3 0
2. MERGE 5 0

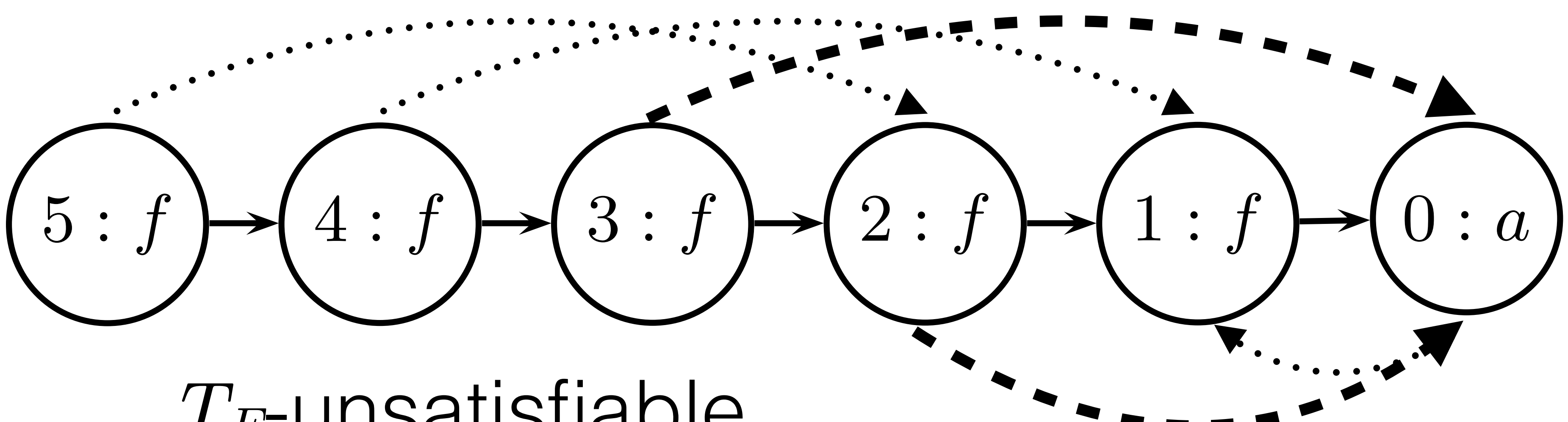


Deciding T_E-Satisfiability Example 2

$$F : f^3(a) = a \land f^5(a) = a \land f(a) \neq a$$

$$S_F = \{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a))\}$$

1. MERGE 3 0
2. MERGE 5 0



T_E-unsatisfiable

Soundness and Completeness

Theorem (Sound & Complete). Quantifier-free conjunctive Σ_E-formula F is T_E-satisfiable iff the congruence closure algorithm returns satisfiable

Complexity

Let e be the number of edges and n be the number of nodes in the initial DAG.

Theorem (Complexity). The congruence closure algorithm run in time $O(e^2)$ for $O(n)$ MERGEs.

Recursive Data Structures

T_{RDS}

- Can model
  - records
  - lists
  - trees
  - stacks
- Cannot model
  - queues

Theory of Lists - T_{cons}

$$\Sigma_{cons} : \{cons,\ car,\ cdr,\ atom,\ =\}$$

- $cons$: a binary function, called the constructor;
- car: a unary function, called the left projector;
- cdr: a unary function, called the right projector;
- $atom$: a unary predicate;
- $=$: a binary predicate

$$car(cons(a,\ b)) = a$$
$$cdr(cons(a,\ b)) = b$$

Axioms of T_{cons}

- Axioms of (reflexivity), (symmetry), and (transitivity) of T_E
- Instantiations of the (function congruence) axiom schema for $cons$, car, and cdr:
  - $\forall x_1, x_2, y_1, y_2.\ x_1 = x_2 \wedge y_1 = y_2 \rightarrow cons(x_1,\ y_1) = cons(x_2,\ y_2)$
  - $\forall x, y.\ x = y \rightarrow car(x) = car(y)$
  - $\forall x, y.\ x = y \rightarrow cdr(x) = cdr(y)$
- An instantiation of the (predicate congruence) axiom schema for atom:
  - $\forall x, y.\ x = y \rightarrow (atom(x) \leftrightarrow atom(y))$

Axioms of T_{cons}

- $\forall x, y.\ car(cons(x,\ y)) = x$ (left projection)
- $\forall x, y.\ cdr(cons(x,\ y)) = y$ (right projection)
- $\forall x.\ \neg atom(x) \rightarrow cons(car(x),\ cdr(x)) = x$ (construction)
- $\forall x, y.\ \neg atom(cons(x,\ y))$ (atom)

Decidability

- T_{cons}: undecidable
- quantifier-free T_{cons}: decidable

Preprocess

By the (construction) axiom, replace

$$\neg atom(u_i)$$

with

$$u_i = cons(u_i^1,\ u_i^2)$$

$$\forall x.\ \neg atom(x) \rightarrow cons(car(x),\ cdr(x)) = x \qquad \text{(construction)}$$

Decision Procedure

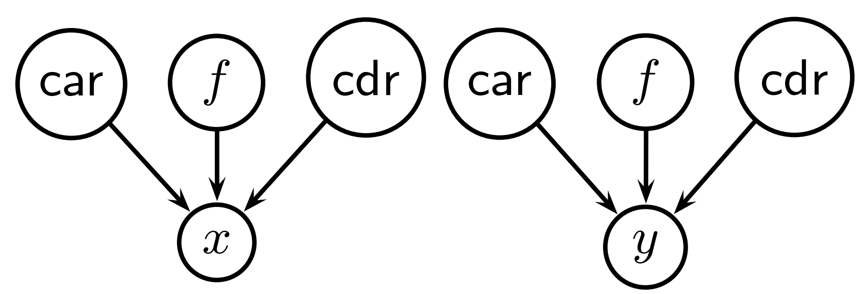
$$F : s_1 = t_1 \wedge \ldots \wedge s_m = t_m \wedge s_{m+1} \neq t_{m+1} \wedge \ldots \wedge s_n \neq t_n$$
$$\wedge\ atom(u_1) \wedge \ldots \wedge atom(u_l)$$

- Construct the initial DAG for the subterm set S_F
- By the (left projection) and (right projection) axioms, for each node n such that $n.\text{fn} = cons$,
  - add $car(n)$ to the DAG and $\text{MERGE}\ car(n)\ \ n.\text{args}[1]$;
  - add $cdr(n)$ to the DAG and $\text{MERGE}\ cdr(n)\ \ n.\text{args}[2]$;
- For $i \in \{1, \ldots, m\}$, $\text{MERGE}\ s_i\ t_i$
- For $i \in \{m + 1, \ldots, n\}$, if $\text{FIND}\ s_i = \text{FIND}\ t_i$, return unsatisfiable
- By the (atom axiom), for $i \in \{1, \ldots, l\}$, if $\exists v.\ \text{FIND}\ v = \text{FIND}\ u_i \wedge v.\text{fn} = cons$, return unsatisfiability
- Otherwise, return satisfiable

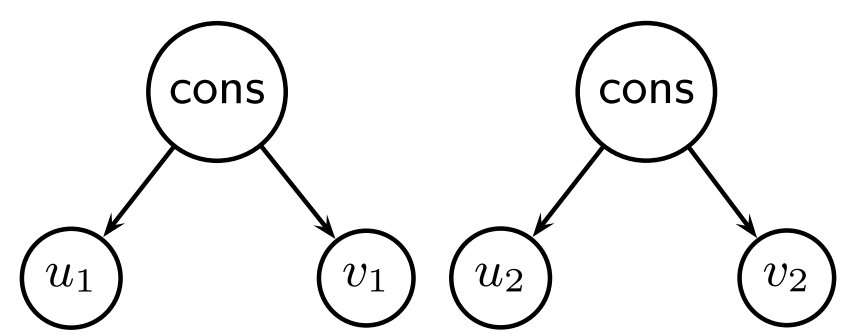
Combining T_E and T_{cons} - Example

$$F: car(x) = car(y) \wedge cdr(x) = cdr(y) \wedge f(x) \neq f(y) \wedge \neg atom(x) \wedge \neg atom(y)$$

$$F': car(x) = car(y) \wedge cdr(x) = cdr(y) \wedge f(x) \neq f(y) \wedge x = cons(u_1, v_1) \wedge y = cons(u_2, v_2)$$

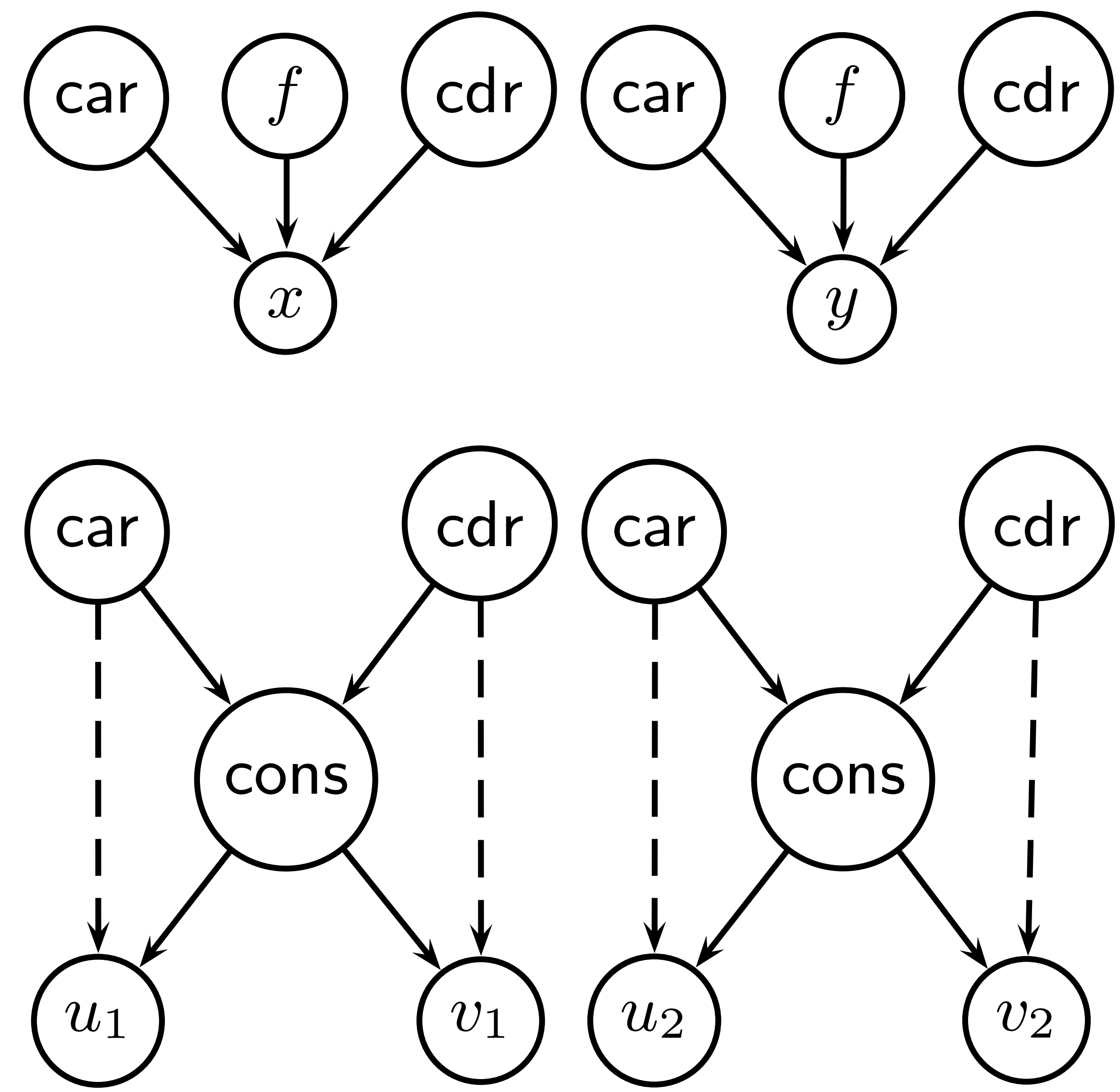


Step 1: initial DAG



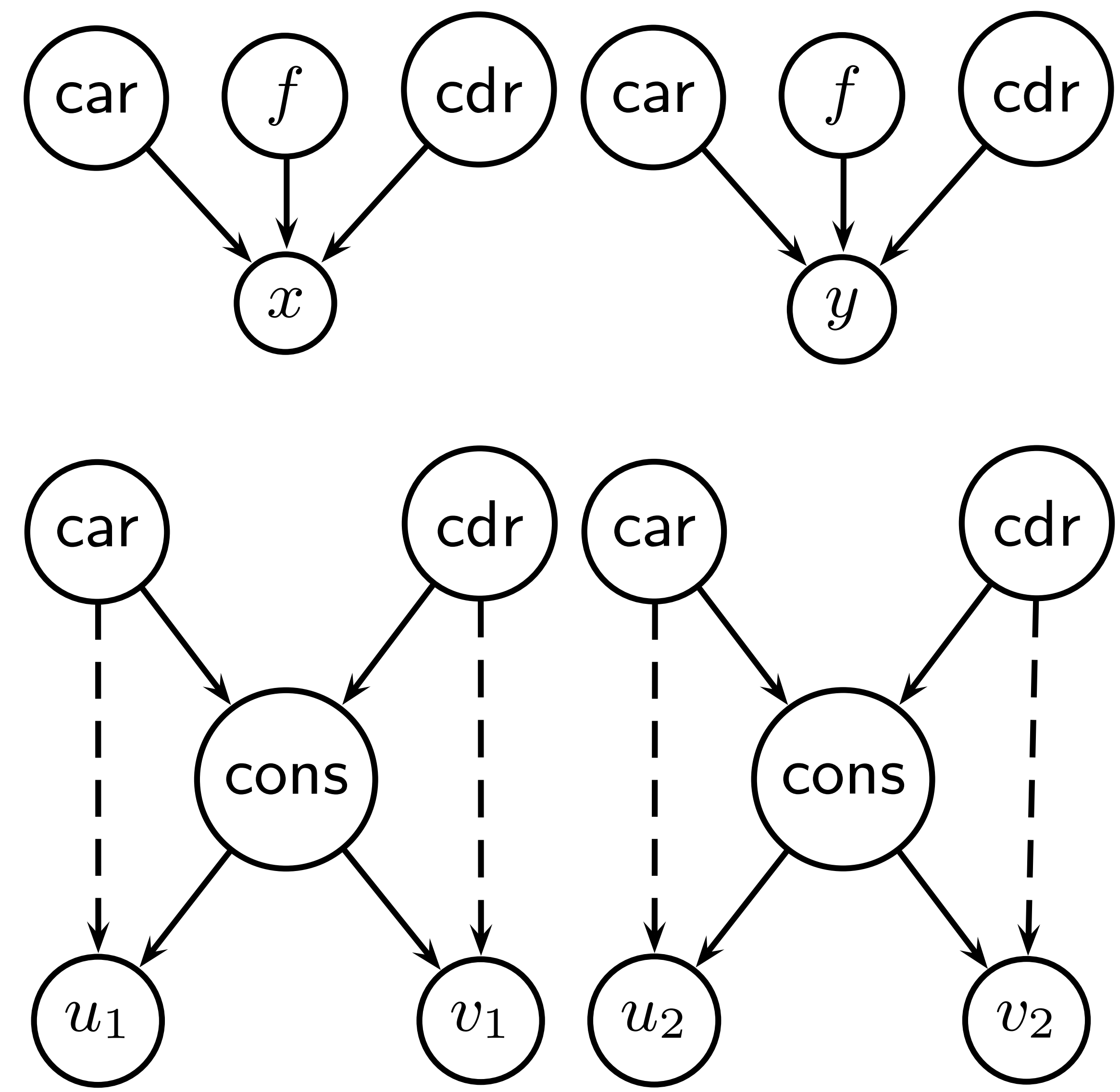
Combining T_E and T_{cons} - Example

Step 2: add $car(n)$ and $cdr(n)$



Combining T_E and T_{cons} - Example

Step 3: $\text{MERGE}\ s_i\ t_i$



Combining T_E and T_{cons} - Example

Step 3: $\mathrm{MERGE}\ s_i\ t_i$

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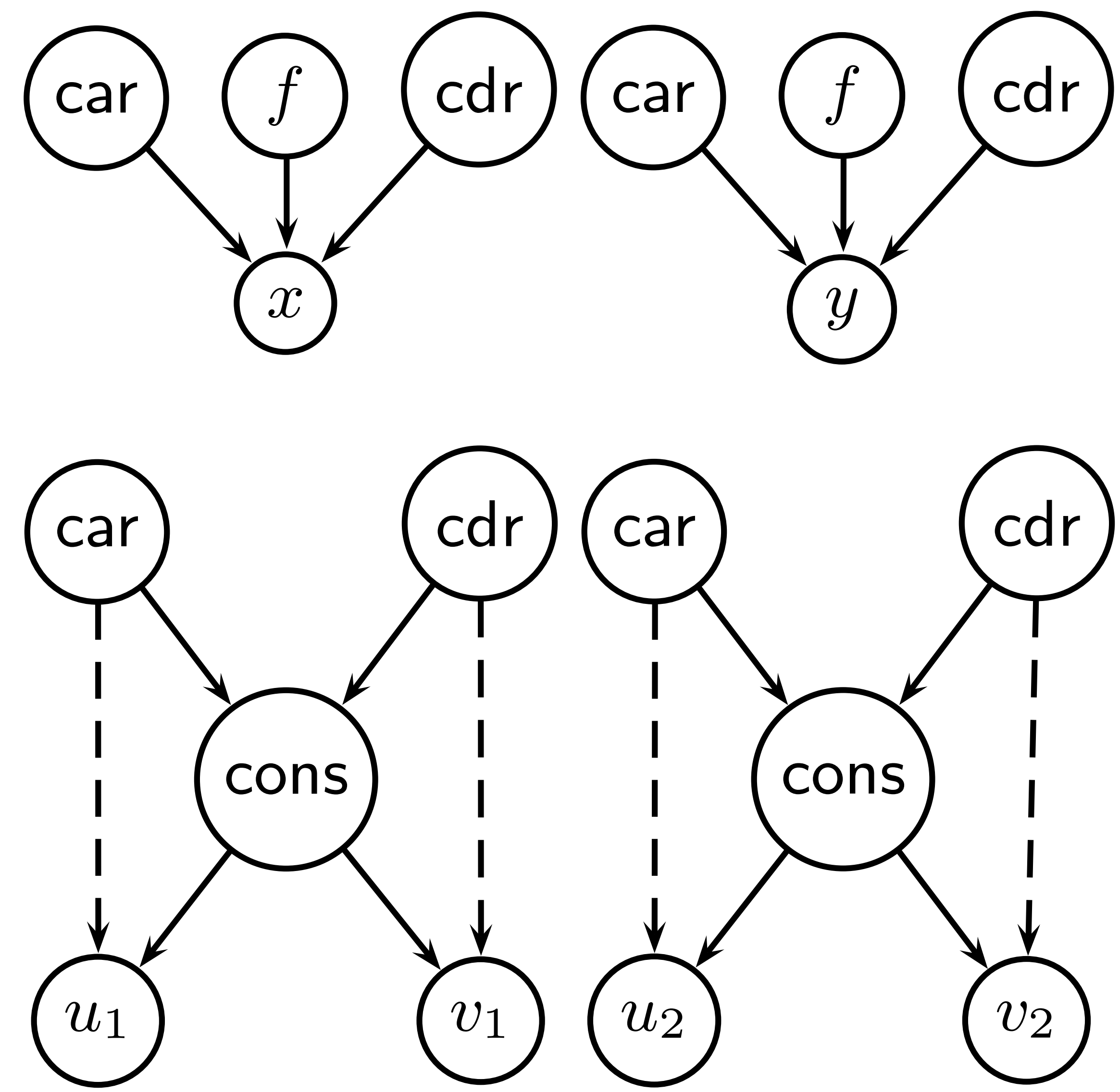

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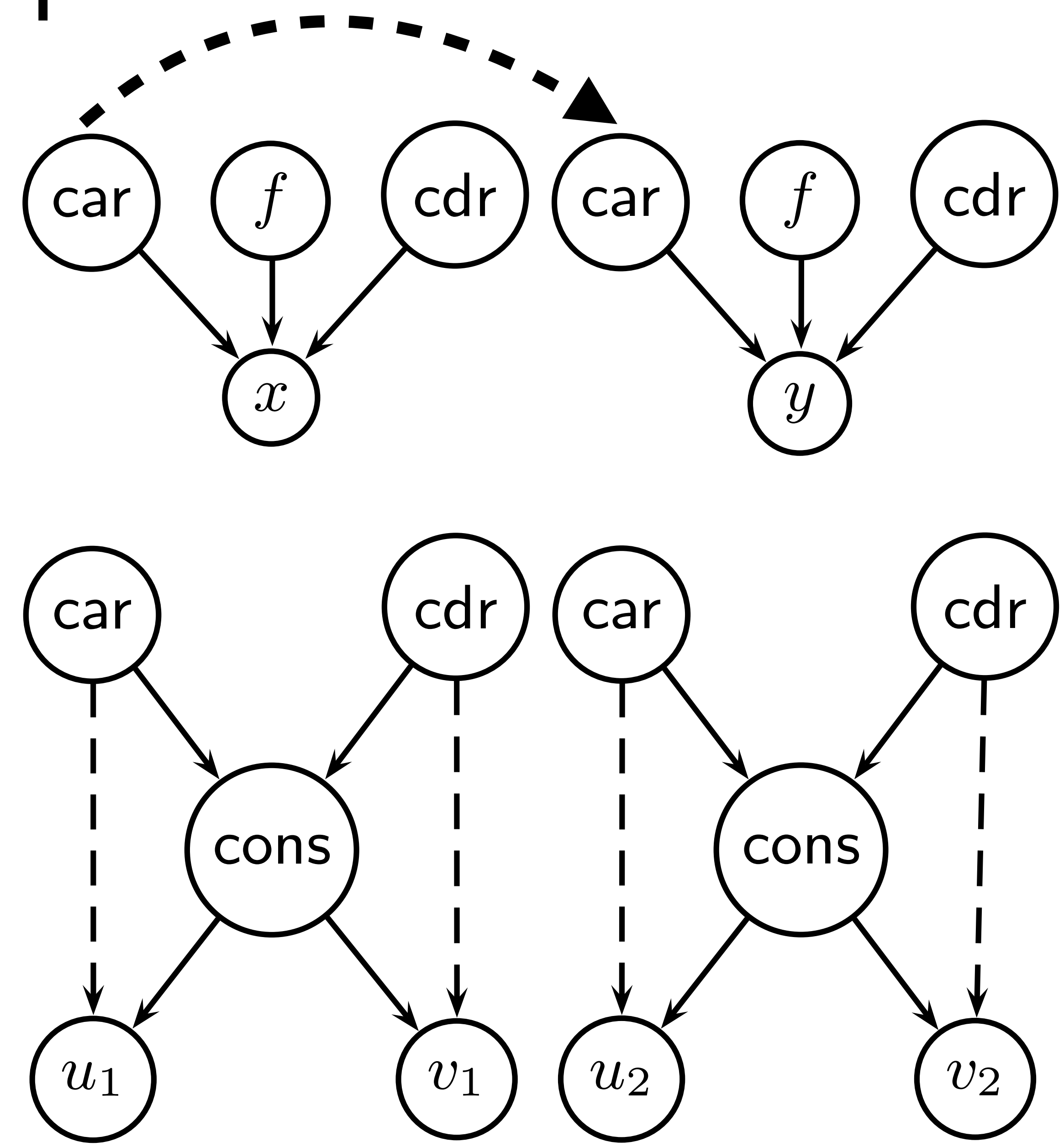

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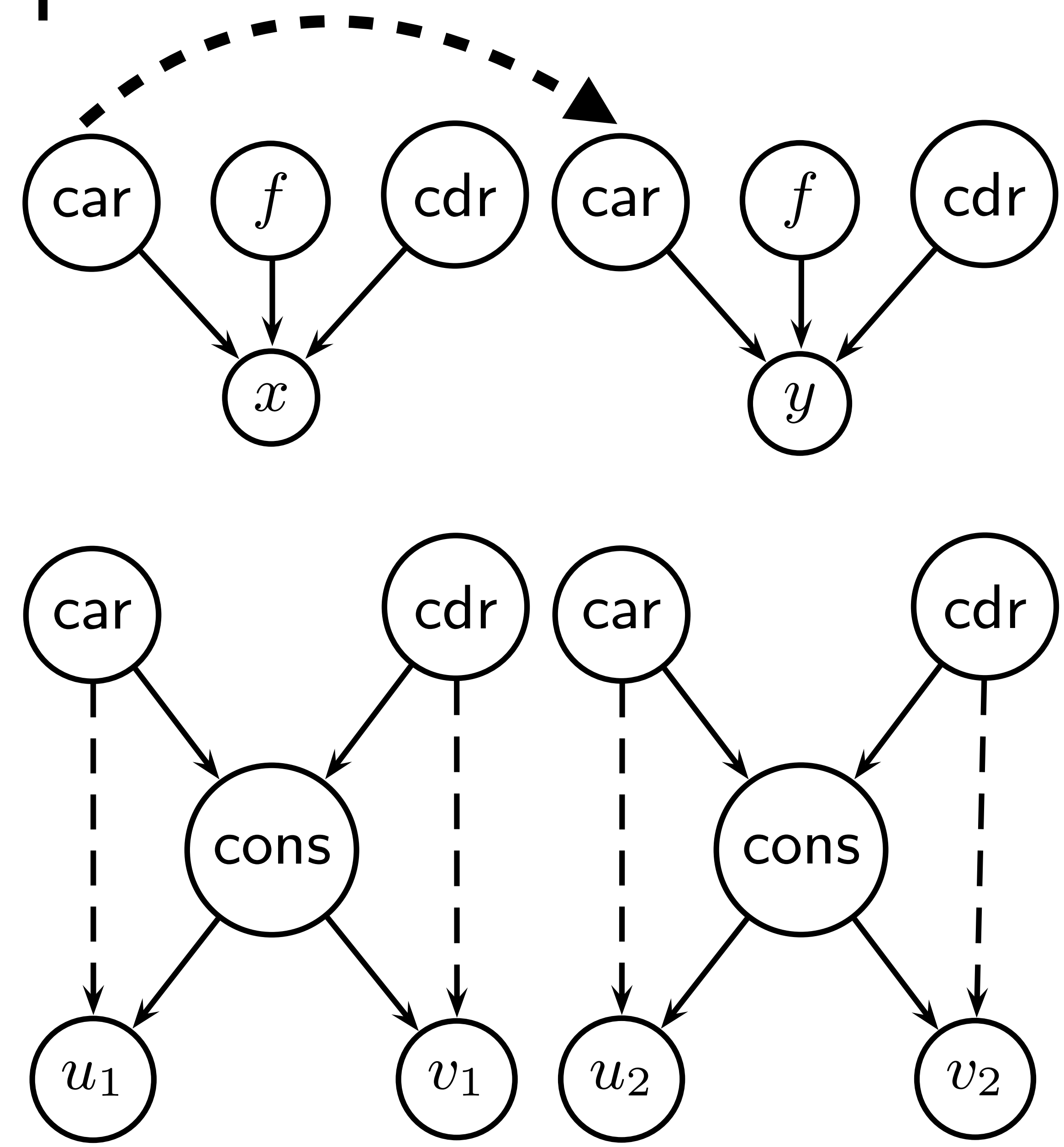

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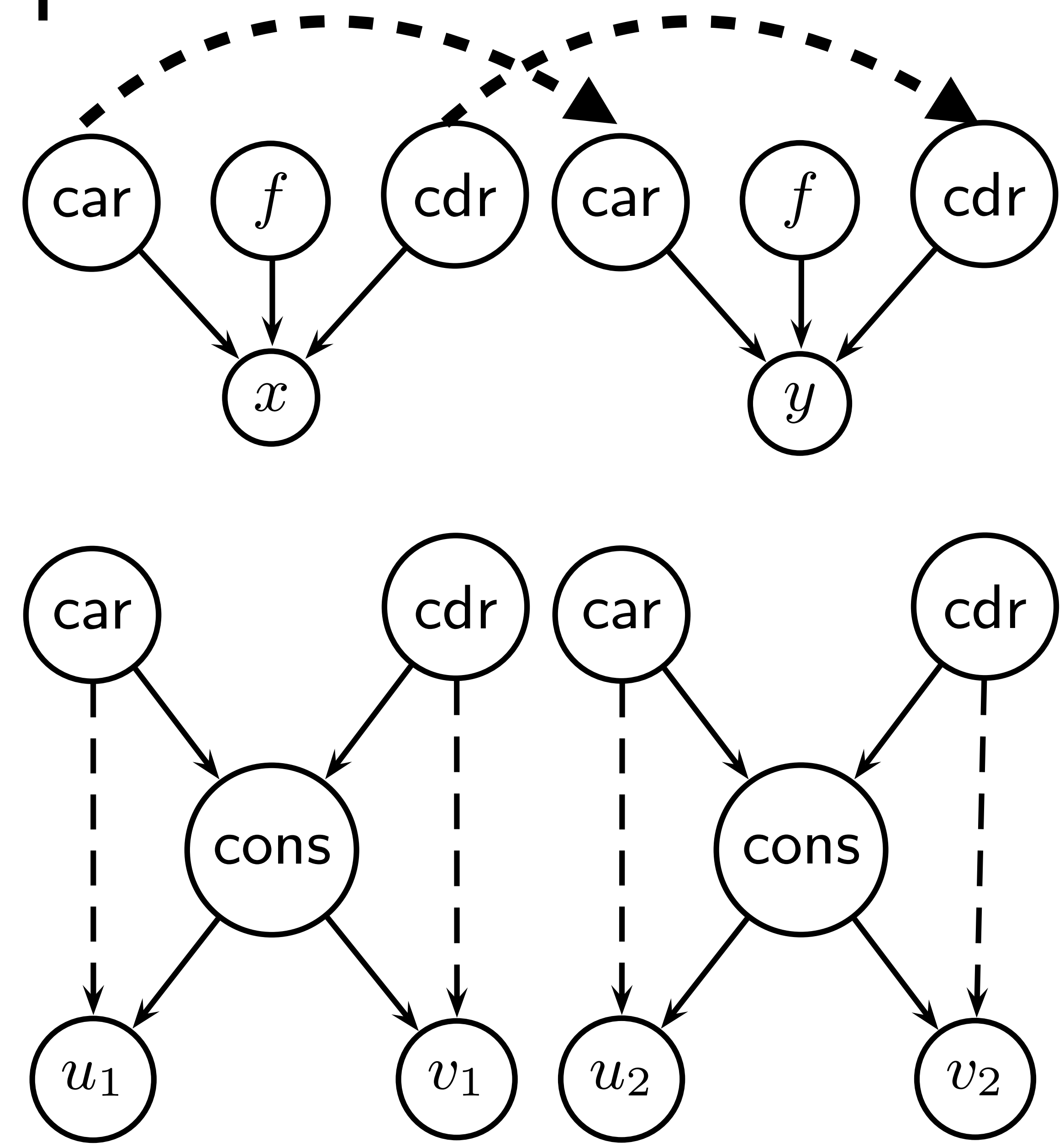

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Step 3: $\text{MERGE } s_i \ t_i$

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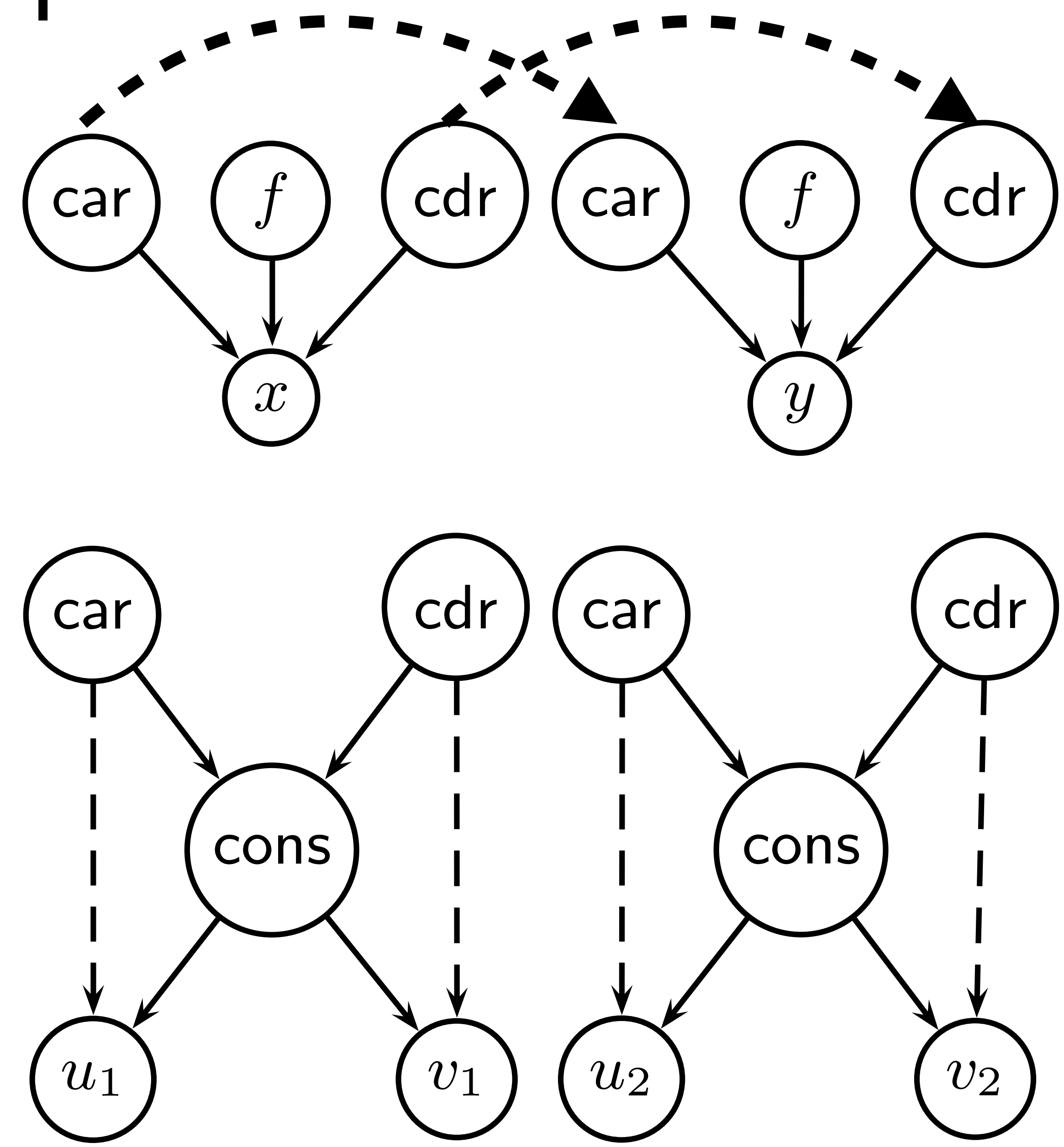

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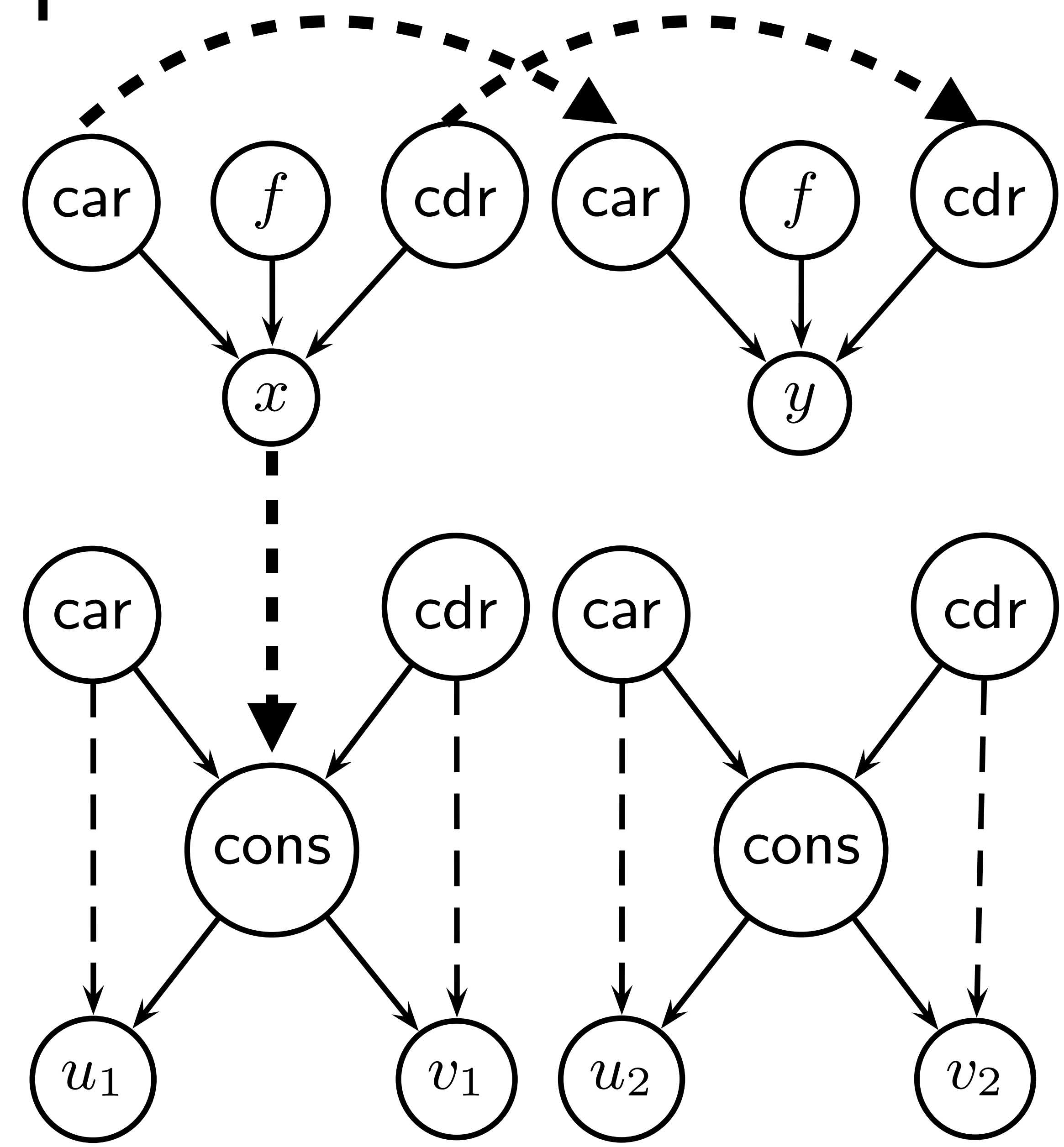

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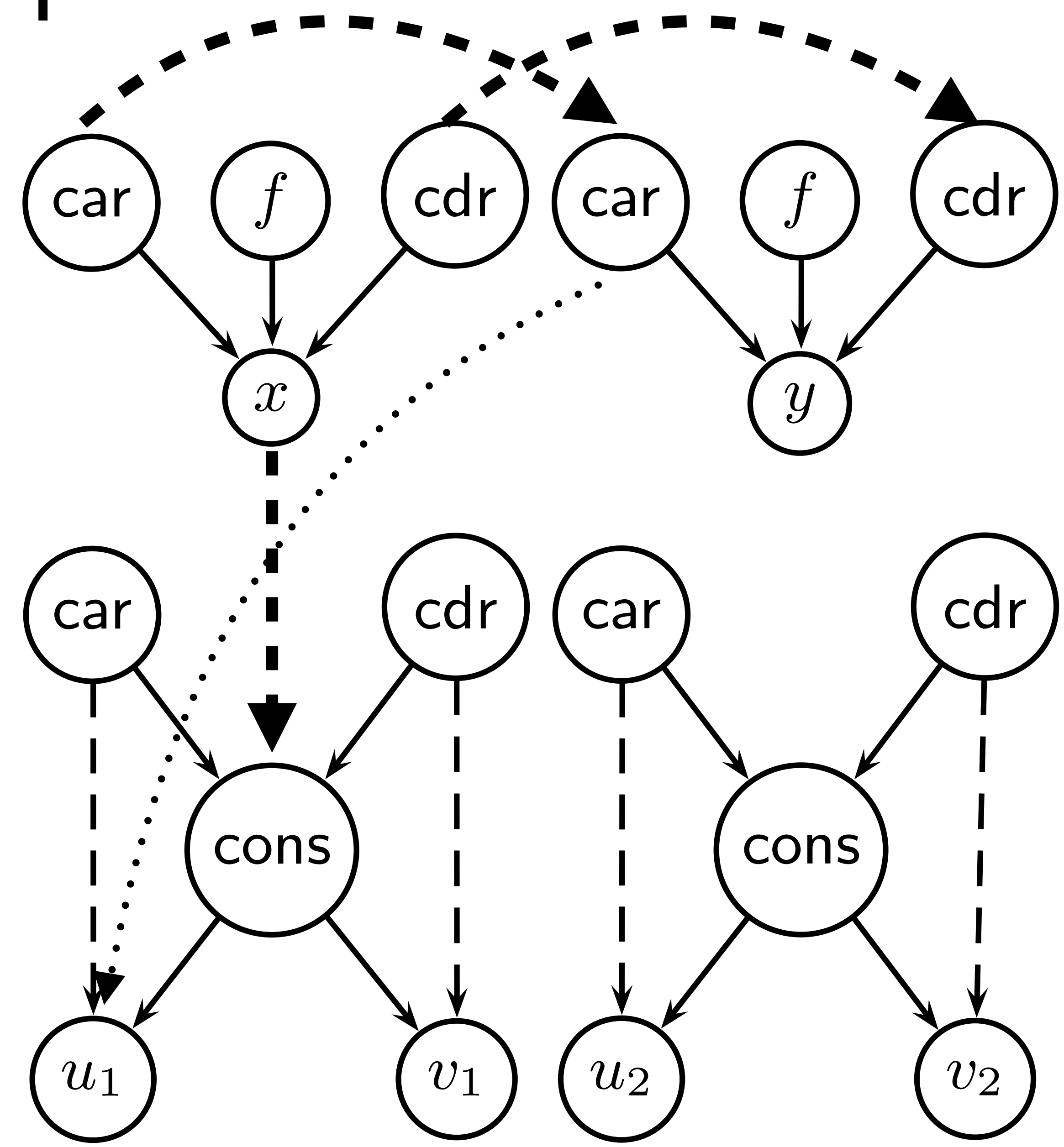

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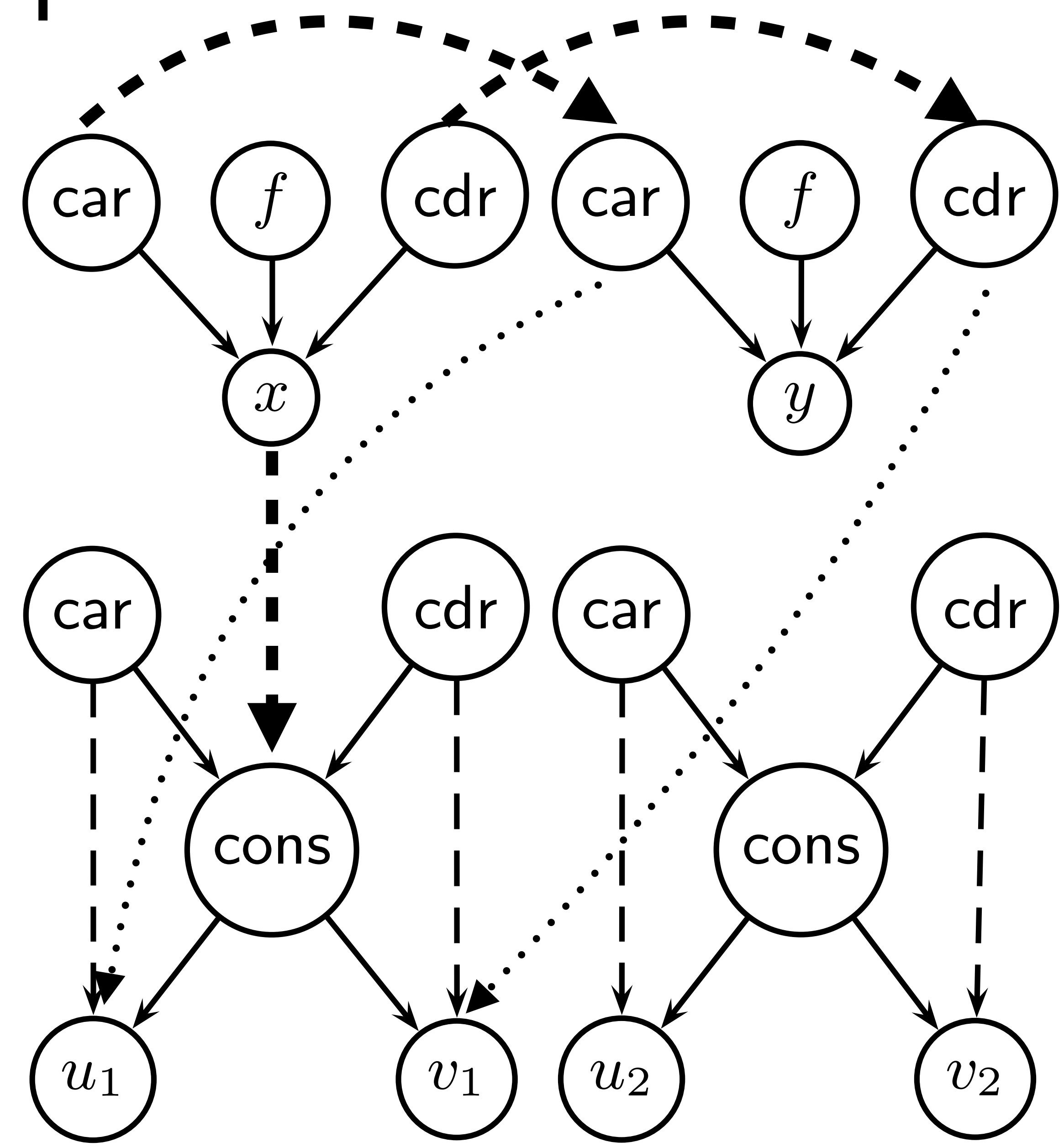

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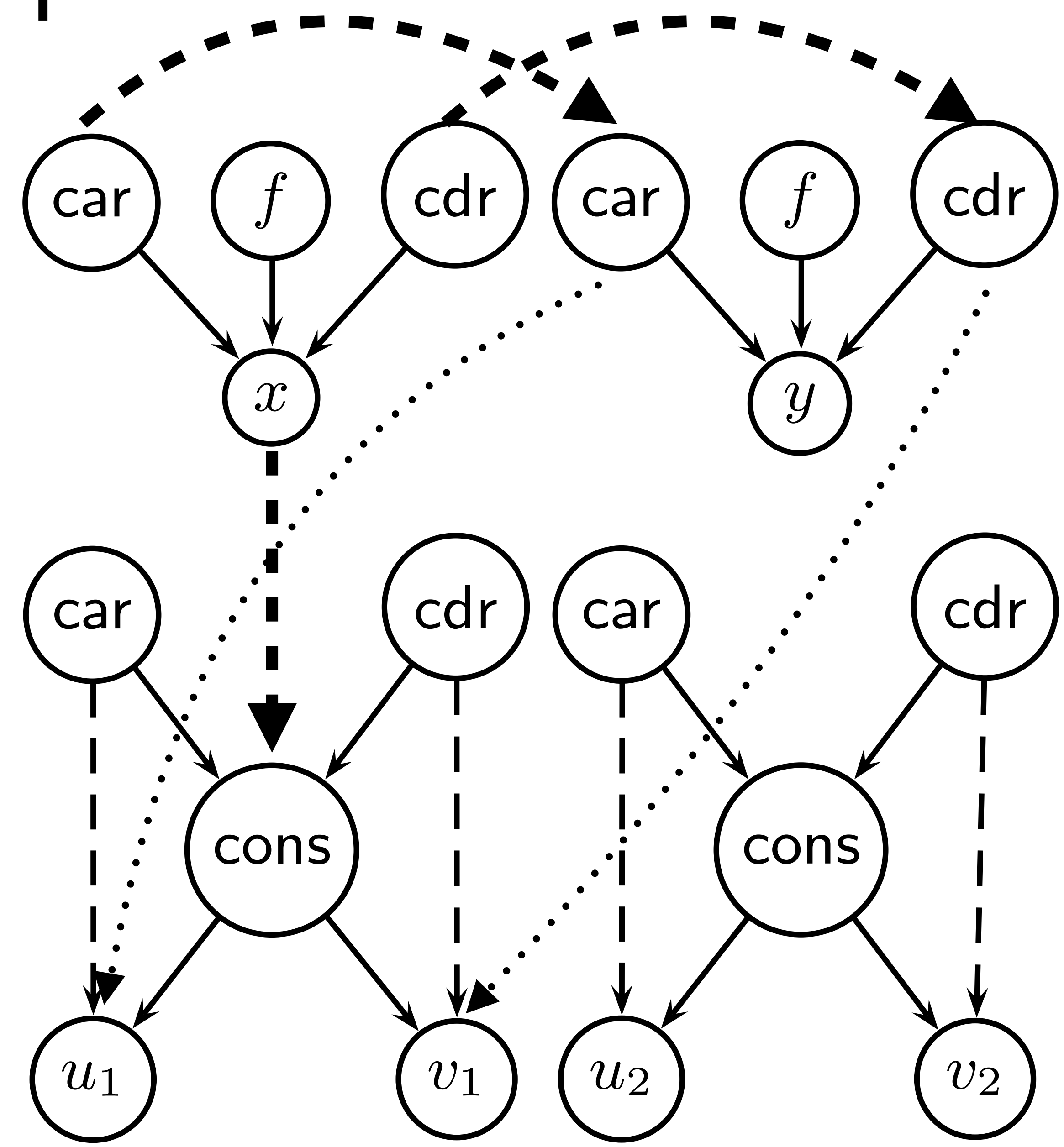

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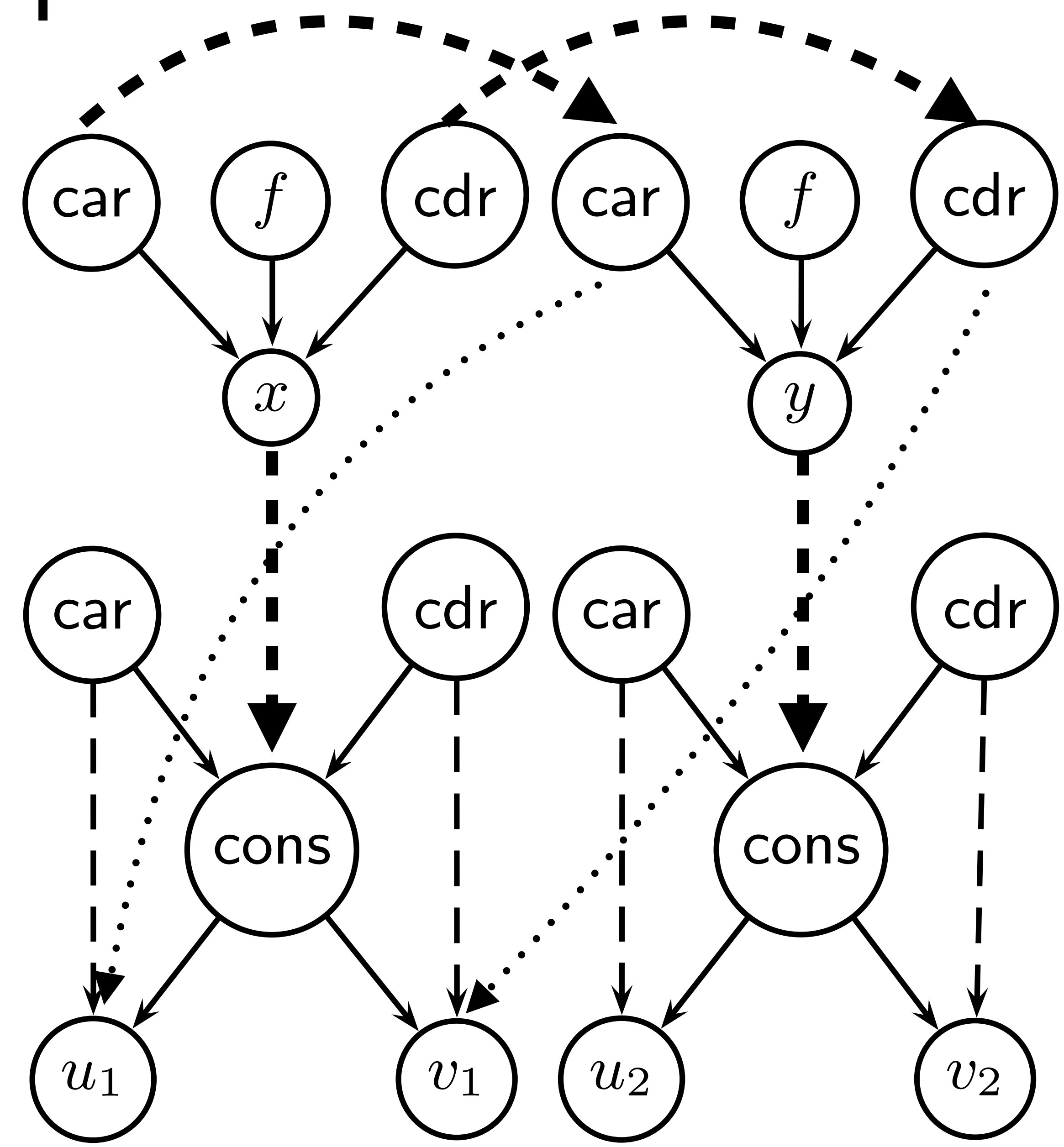

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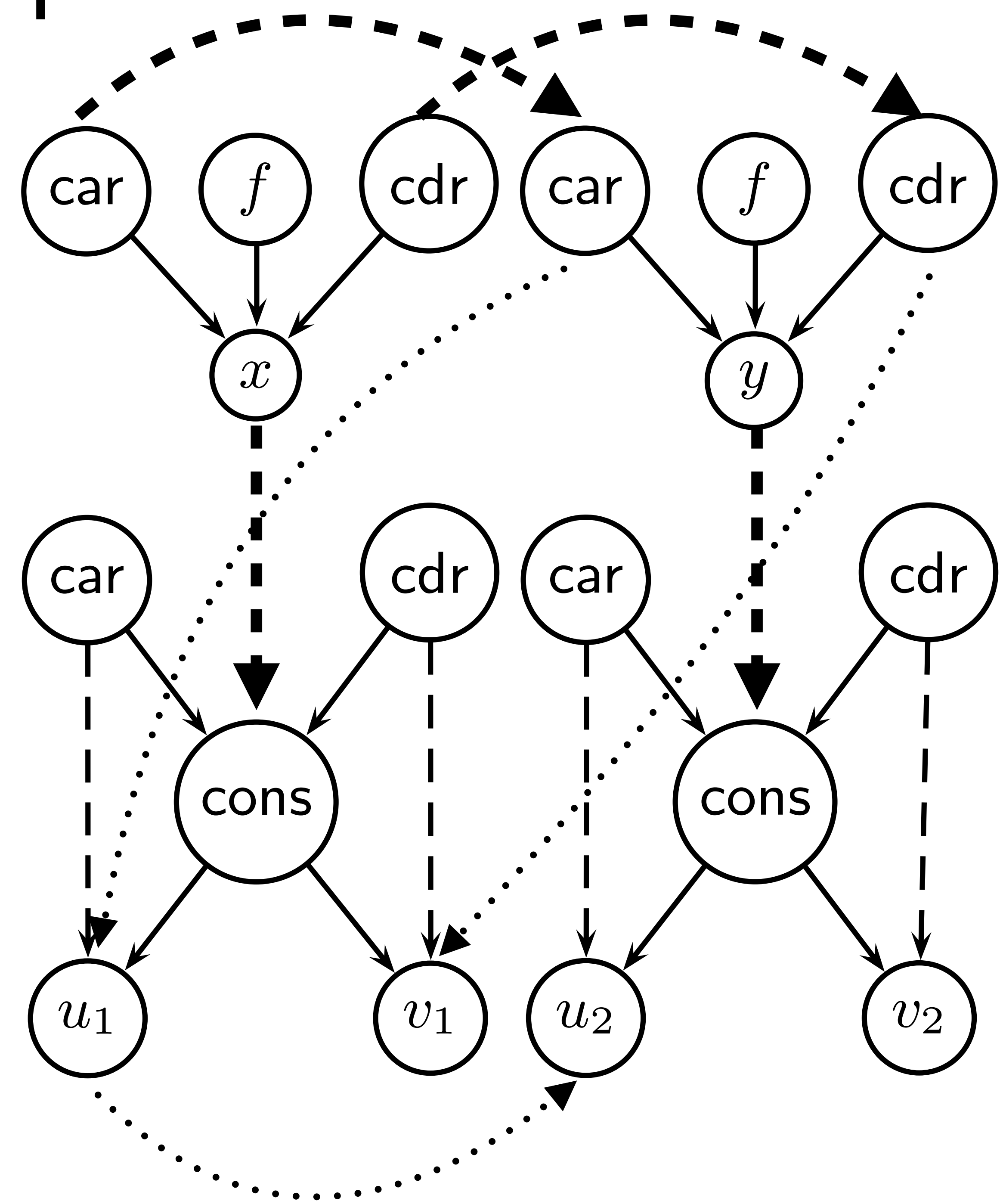

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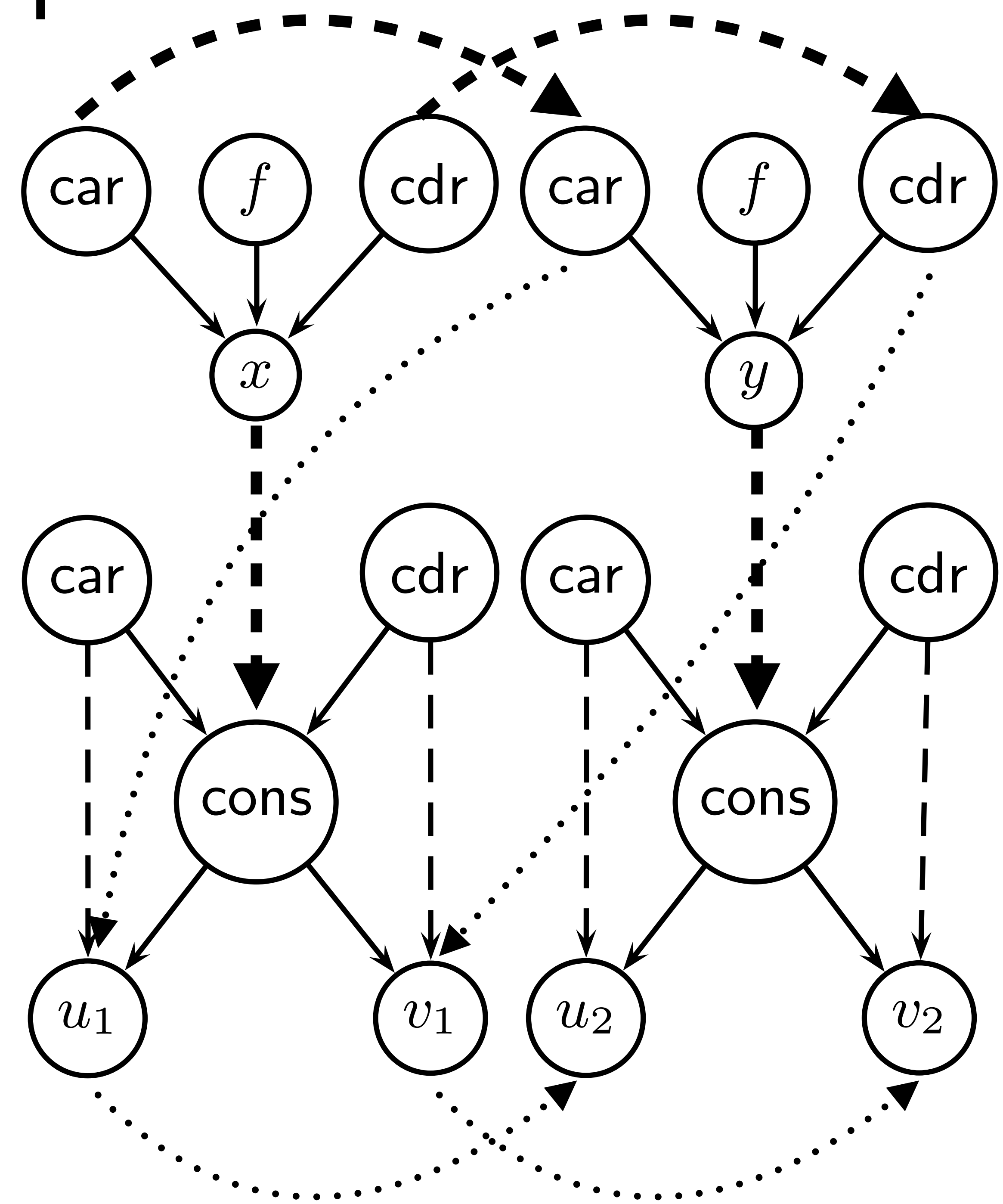

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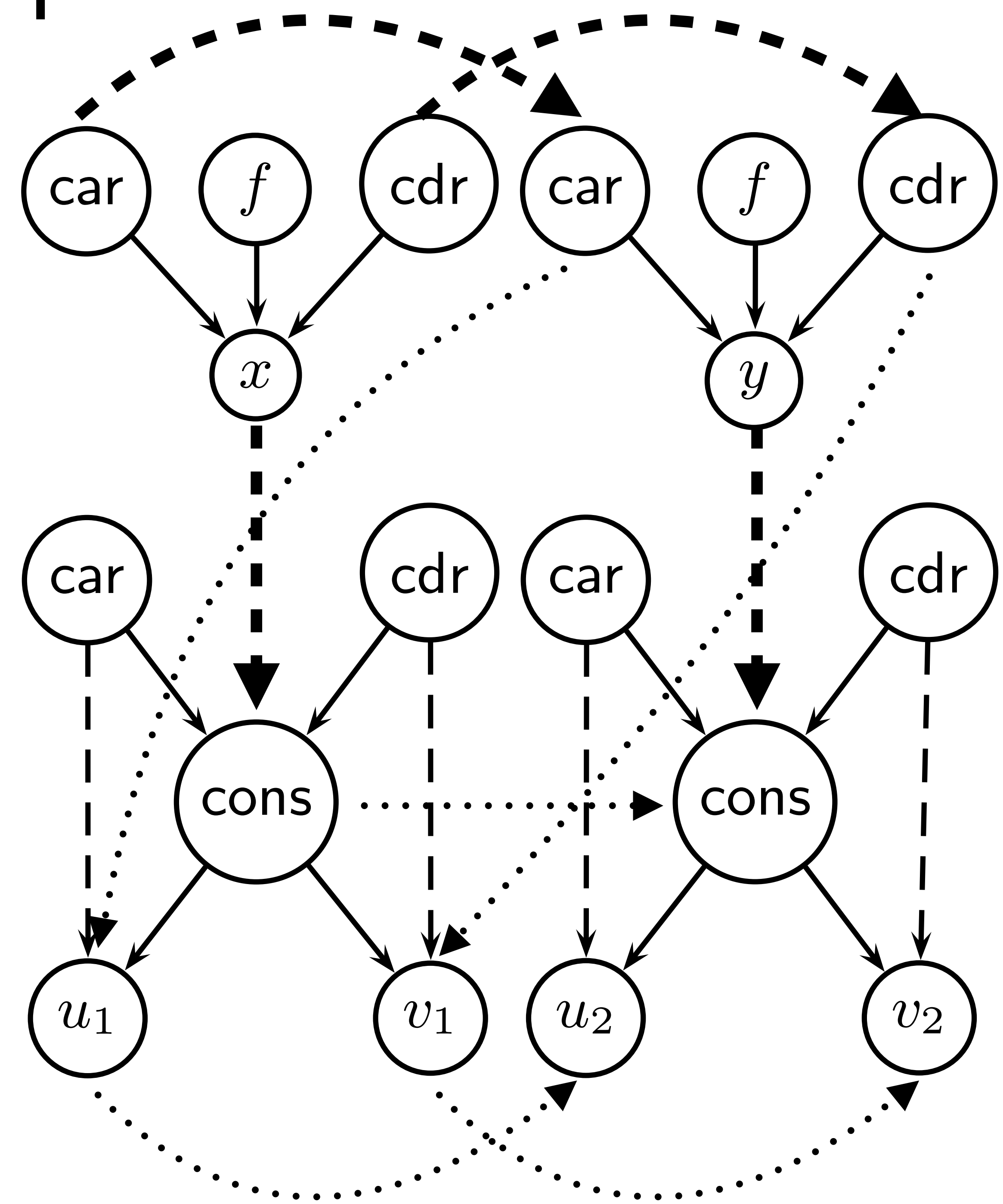

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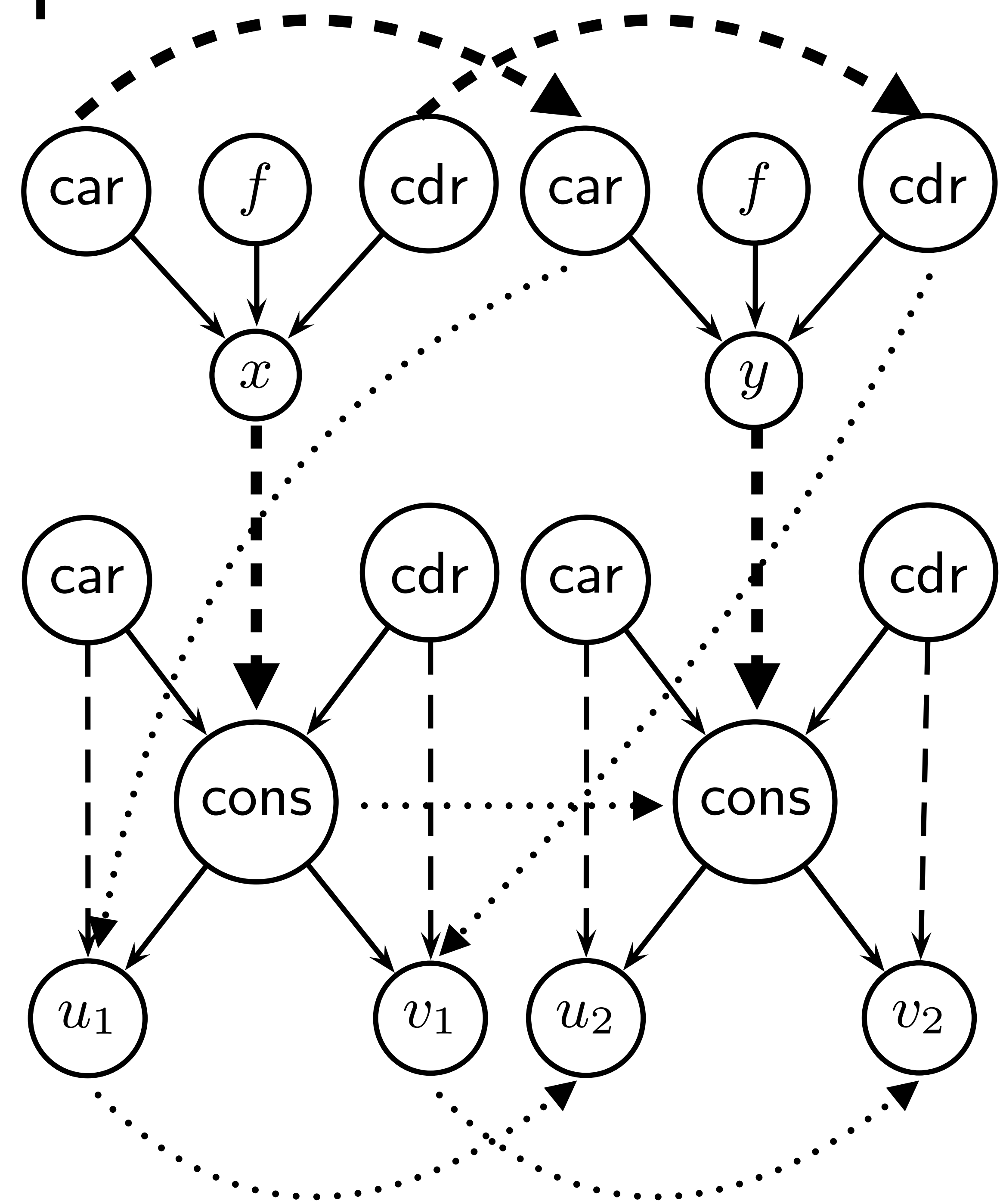
$(T_{cons} \cup T_E)$-unsatisfiable



Arrays

Theory of Arrays - T_A

$$\Sigma_{\mathrm{A}} : \{\cdot[\cdot],\ \cdot\langle\cdot \triangleleft \cdot\rangle,\ =\}$$

- $a[i]$: a binary function; $a[i]$ represents the value of array a at position i;
- $a\langle i \triangleleft v\rangle$: a ternary function; $a\langle i \triangleleft v\rangle$ represents the modified array a in which position i has value v;
- $=$: a binary predicate

Axioms of T_A

- Axioms of (reflexivity), (symmetry), and (transitivity) of T_E
- $\forall a, i, j.\ i = j \rightarrow a[i] = a[j]$ (array congruence)
- $\forall a, v, i, j.\ i = j \rightarrow a\langle i \triangleleft v \rangle[j] = v$ (read-over-write 1)
- $\forall a, v, i, j.\ i \neq j \rightarrow a\langle i \triangleleft v \rangle[j] = a[j]$ (read-over-write 2)

Decision Procedure

- Based on a reduction to T_E-satisfiability via applications of the (read-over-write) axioms
- If the formula does not contain any write terms, then the read terms can be viewed as uninterpreted function terms
- Otherwise, any write term must occur in the context of a read

Decision Procedure - Step 1

If F does not contain any write terms $a\langle i \triangleleft v\rangle$, perform the following steps.

1. Associate each array variable a with a fresh function symbol f_a, and replace each read term $a[i]$ with $f_a(i)$

2. Decide and return the T_E-satisfiability of the resulting formula

Decision Procedure - Step 2

Select some read-over-write term $a\langle i \triangleleft v\rangle[j]$, and split on two cases:

1. According to (read-over-write 1), replace

   $F[a\langle i \triangleleft v\rangle[j]]$ with F_1: $F[v] \wedge i = j$

   and recurse on F_1. If F_1 is found to be T_A-satisfiable, return satisfiable

2. According to (read-over-write 2), replace

   $F[a\langle i \triangleleft v\rangle[j]]$ with F_2: $F[a[j]] \wedge i \neq j$

   and recurse on F_2. If F_2 is found to be T_A-satisfiable, return satisfiable

If both F_1 and F_2 are found to be T_A-unsatisfiable, return unsatisfiable

Example of T_A

$F: i_1 = j \land i_1 \neq i_2 \land a[j] = v_1 \land a\langle i_1 \triangleleft v_1\rangle\langle i_2 \triangleleft v_2\rangle[j] \neq a[j]$

- First case:
  - F_1: $i_2 = j \land i_1 = j \land i_1 \neq i_2 \land a[j] = v_1 \land v_2 \neq a[j]$
  - F_1': $i_2 = j \land i_1 = j \land i_1 \neq i_2 \land f_a(j) = v_1 \land v_2 \neq f_a(j)$
  - F_1 is T_A-unsatisfiable

Example of T_A

$F : i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge a\langle i_1 \triangleleft v_1 \rangle\langle i_2 \triangleleft v_2 \rangle[j] \neq a[j]$

- Second case:
    - F_2: $i_2 \neq j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge a\langle i_1 \triangleleft v_1 \rangle[j] \neq a[j]$
        - F_3: $i_1 = j \wedge i_2 \neq j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge v_1 \neq a[j]$
        - F_4: $i_1 \neq j \wedge i_2 \neq j \wedge i_1 = j \wedge i_1 \neq i_2 \wedge a[j] = v_1 \wedge a[j] \neq a[j]$
    - F_2 is T_A-unsatisfiable

Soundness and Completeness

Theorem (Sound & Complete). Given quantifier-free conjunctive Σ_A-formula F, the decision procedure returns satisfiable iff F is T_A-satisfiable; otherwise, it returns unsatisfiable

Complexity

Theorem (Complexity). T_A-satisfiability of quantifier-free conjunctive Σ_A-formula is NP-complete

Summary

- Congruence closure algorithm
    - relations, equivalence relations, congruence relations, partitions, quotients, classes, closures
- DAG-based implementation
    - union-find, merge
- Recursive data structures
    - T_{cons}
- Arrays