Quantifier-Free Linear Arithmetic

Bow-Yaw Wang

Institute of Information Science
Academia Sinica, Taiwan

June 23, 2015

Outline

Quantifier-Free Fragment of $T_{\mathbb{Q}}$

- Let $T_{\mathbb{Q}}$ denote the theory of rationals.
- We consider only *linear* constraints in this lecture.
  - That is, propositions are of the form

$$\sum_{i=1}^{n} a_i x_i \approx b$$

  where $a_i, b \in \mathbb{Q}$ and $\approx \in \{<, \leq\}$.
- The *quantifier-free fragment* of $T_{\mathbb{Q}}$ considers formulae of the form:

$$G : \forall x_1, x_2, \ldots, x_n . F[x_1, x_2, \ldots, x_n]$$

  where $x_1, x_2, \ldots, x_n$ are rational variables, F has no quantifiers with free variables $x_1, x_2, \ldots, x_n$.
- We want to decide if G holds, equivalently, F is $T_{\mathbb{Q}}$-*valid*.

Validity and Unsatisfiability

- Observe that

$$G : \forall x_1, x_2, \ldots, x_n . F[x_1, x_2, \ldots, x_n]$$

  and

$$\neg G : \exists x_1, x_2, \ldots, x_n . \neg F[x_1, x_2, \ldots, x_n]$$

  are equivalent.
- F is $T_{\mathbb{Q}}$-valid iff $\neg F$ is $T_{\mathbb{Q}}$-unsatisfiable.
- We thus only consider $T_{\mathbb{Q}}$-satisfiability in this lecture.
  - Note that both validity and satisfiability are needed if there are quantifier alternations such as $\forall x \exists y$ or $\exists x \forall y$.

Conjunctive Quantifier-Free $T_{\mathbb{Q}}$-Formulae

- Recall that any propositional logic formula can be transformed to disjunctive normal form (DNF).
- To further simplify the problem, we only consider conjunctive quantifier-free formulae:

$$
\begin{array}{rl}
 & \sum_{j=1}^{n} a_{1j} x_j \approx_1 b_1 \\
\wedge & \sum_{j=1}^{n} a_{2j} x_j \approx_2 b_2 \\
 & \cdots \\
\wedge & \sum_{j=1}^{n} a_{mj} x_j \approx_m b_m
\end{array}
$$

where $a_{ij}, b_i \in T_{\mathbb{Q}}$ and $\approx_i \in \{<, \leq\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Linear Algebra

- It is too tedious to write conjunctive quantifier-free $T_{\mathbb{Q}}$-formulae.
- We will review and use notations from linear algebra.

Vector and Matrix

- An *n-vector* is a column $\bar{a} \in \mathbb{Q}^n$. The *transpose* $\bar{a}^T$ is a row of the same ordered elements:

$$\bar{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \bar{a}^T = [a_1 \; a_2 \; \cdots \; a_n]$$

- Similarly, we use $\bar{x}$ for a *variable n-vector* with variables $x_1, x_2, \ldots, x_n$.
- An $m \times n$*-matrix* $A \in \mathbb{Q}^{m \times n}$ and its *transpose* $A^T \in \mathbb{Q}^{n \times m}$ are defined similarly:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ & & \vdots & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Row and Column Vectors of Matrix

- Let $A \in \mathbb{Q}^{m\times n}$ be an $m \times n$-matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A *row* $\bar{a}_i$ of A is the row vector

$$\bar{a}_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

- A *column* $\bar{a}_j$ of A is the column vector

$$\bar{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = \begin{bmatrix} a_{1j} & a_{2j} & \cdots & a_{mj} \end{bmatrix}^T$$

Vector-Vector Multiplication

- Let $\bar{a}, \bar{b} \in \mathbb{Q}^{1\times n}$ be vectors.
- Define

$$\bar{a}\bar{b}^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i b_i .$$

Matrix-Vector Multiplication

- Let $A \in \mathbb{Q}^{m \times n}$ and $\bar{x} \in \mathbb{Q}^{n \times 1}$.
- Define

$$A\bar{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i} x_i \\ \sum_{i=1}^{n} a_{2i} x_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi} x_i \end{bmatrix}$$

Matrix-Matrix Multiplication

- Let $A \in \mathbb{Q}^{m \times n}$ and $B \in \mathbb{Q}^{n \times \ell}$.
- Define

$$\begin{aligned} AB &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix} \\ &= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1\ell} \\ c_{21} & c_{22} & \cdots & c_{2\ell} \\ & & \vdots & \\ c_{m1} & c_{m2} & \cdots & c_{m\ell} \end{bmatrix} \end{aligned}$$

where

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

for $1 \leq i \leq m$ and $1 \leq k \leq \ell$.

Useful Notations

- $\bar{0}$ is a column vector of 0's.
- $\bar{1}$ is a column vector of 1's.
  - Hence $\bar{0}^T\bar{x} = 0$ and $\bar{1}^T\bar{x} = \sum_{i=1}^{n} x_i$ when $\bar{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$
- Define the *identity* matrix

$$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

  - Hence $IA = AI = A$ for any $A \in \mathbb{Q}^{n \times n}$.
- Finally, define the *unit* vector

$$e_i = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}^T$$

where all elements are 0 except the i-th position.

Linear Equations

- Consider the conjunctive quantifier-free linear $T_{\mathbb{Q}}$-formula:

$$\sum_{j=1}^{n} a_{1j}x_j = b_1 \bigwedge \sum_{j=1}^{n} a_{2j}x_j = b_2 \bigwedge \cdots \bigwedge \sum_{j=1}^{n} a_{mj}x_j = b_m$$

- Define $A \in \mathbb{Q}^{m \times n}$, $\bar{x} \in \mathbb{Q}^{n \times 1}$, and $\bar{b} \in \mathbb{Q}^{m \times 1}$ as follows.

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ \bar{x} &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T \\ \bar{b} &= \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix}^T \end{aligned}$$

- Then $A\bar{x} = \bar{b}$ describes the same quantifier-free formula.

Solving Linear Equations I

- $A\bar{x} = \bar{b}$ can be solved by *elementary row operations*:
  - Swap two rows.
  - Multiply a row by a nonzero scalar.
  - Add one row to another.
- It sounds a bit scary but it really is high school mathematics.

Solving Linear Equations II

- Let's solve

$$\begin{array}{ccccccc} 3x_1 & + & x_2 & + & 2x_3 & = & 6 \\ x_1 & & & + & x_3 & = & 1 \\ 2x_1 & + & 2x_2 & + & x_3 & = & 2 \end{array}$$

with high school mathematics (but new notation):

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 6 \\ 1 & 0 & 1 & 1 \\ 2 & 2 & 1 & 2 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 3 & 1 & 2 & 6 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 0 & 1 & -1 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 2 & -1 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -6 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

- That is, $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -6 \end{bmatrix}$.

Linear Inequality

- Let $A \in \mathbb{Q}^{m \times n}$, $\bar{x}$ a variable n-vector, and $\bar{b}$ an m-vector.
- The *linear inequality*
$$G : \quad A\bar{x} \leq \bar{b}$$
represents
$$G : \quad \bigwedge_{i=1}^{m} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$
- The set of n-dimensional points described by a linear inequality is called a *polyhedron*.

Convexity of Polyhedron

- A set $S \subseteq R^n$ is *convex* if for every $\bar{v}_1, \bar{v}_2 \in S$,

$$\lambda \bar{v}_1 + (1 - \lambda)\bar{v}_2 \in S \text{ for } \lambda \in [0, 1].$$

Theorem

Any polyhedron is convex.

Proof.

Let $A\bar{x} \leq \bar{b}$ defines a polyhedron P, and $\bar{v}_1, \bar{v}_2 \in P$. Then

$$A\bar{v}_1 \leq \bar{b} \text{ and } A\bar{v}_2 \leq \bar{b}.$$

Let $\lambda \in [0, 1]$. We have $A\lambda\bar{v}_1 \leq \lambda\bar{b}$ and $A(1 - \lambda)\bar{v}_2 \leq (1 - \lambda)\bar{b}$. Hence $A(\lambda\bar{v}_1 + (1 - \lambda)\bar{v}_2) = A\lambda\bar{v}_1 + A(1 - \lambda)\bar{v}_2 \leq \lambda\bar{b} + (1 - \lambda)\bar{b} = \bar{b}$. □

Vertex and Adjacent Vertices

- Let $A \in \mathbb{Q}^{m \times n}$, $\bar{x}$ a variable n-vector, and $\bar{b}$ an m-vector.
- Consider the linear inequality $A\bar{x} \leq \bar{b}$.
- An n-vector $\bar{v}$ is a *vertex* of $A\bar{x} \leq \bar{b}$ if there is an $n \times n$-submatrix A_0 of A and corresponding n-subvector $\bar{b}_0$ of $\bar{b}$ such that $A_0\bar{v} = \bar{b}_0$.
  - The rows in A_0 and $\bar{b}_0$ are called the *defining constraints* of $\bar{v}$.
- Two vertices are *adjacent* if their defining constraints differ by only one constraint.

Linear Program

- Let $A \in \mathbb{Q}^{m \times n}$, $\bar{x}$ a variable n-vector, $\bar{c}$ an n-vector, and $\bar{b}$ an m-vector.
- The *linear program* (or *linear optimization problem*)

$$\begin{array}{ll} \textbf{max} & \bar{c}^T \bar{x} \\ \textbf{subject to} & \\ & A\bar{x} \leq \bar{b} \end{array}$$

  is solved by an n-vector $\bar{v}^*$ which satisfies $A\bar{v}^* \leq \bar{b}$ with the maximal value $\bar{c}^T \bar{v}^*$.
  - $\bar{c}^T \bar{x}$ is the *objective function*.
  - $A\bar{x} \leq \bar{b}$ is the *constraints*.
- If $A\bar{x} \leq \bar{b}$ is unsatisfiable, the maximum is $-\infty$ by convention.
- If the maximum is unbounded, we say the maximum is ∞ by convention.

Example I

- Consider

$$\textbf{max} \quad \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix}$$

$$\textbf{subject to}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

- $x + y - z_1 - z_2$ is the objective function.

Example II

- $\begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix}^T$ is a vertex because of rows 3, 4, 5, 6.

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \end{bmatrix}$$

- Also, $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ is a vertex. (why?)

Duality Theorem

Theorem

Let $A \in \mathbb{Q}^{m \times n}$, $\bar{b}$ an m-vector, and $\bar{c}$ an n-vector. If $A\bar{x} \leq \bar{b}$ is satisfiable, then

$$\max\{\bar{c}^T\bar{x} : A\bar{x} \leq \bar{b}\} = \min\{\bar{y}^T\bar{b} : \bar{y} \geq 0 \textit{ and } \bar{y}^T A = \bar{c}^T\}.$$

- $\bar{x}$ ia a variable n-vector and $\bar{y}$ is a variable m-vector.
- Observe that the right hand side is a system of equality.

$T_{\mathbb{Q}}$ Satisfiability I

- In $T_{\mathbb{Q}}$, two inequalities ($<$ and $\leq$) are allowed.
- Linear programming only allows $\leq$.
- Consider

$$
\begin{array}{rcl}
F: & & \bigwedge_{i=1}^{m} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i \\
& \wedge & \bigwedge_{j=1}^{\ell} a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n < \beta_j
\end{array}
$$

and

$$
\begin{array}{rcl}
F': & & \bigwedge_{i=1}^{m} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i \\
& \wedge & \bigwedge_{j=1}^{\ell} a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n + x_{n+1} \leq \beta_j \\
& \wedge & x_{n+1} > 0
\end{array}
$$

- F is $T_{\mathbb{Q}}$-satisfiable iff F' is $T_{\mathbb{Q}}$-satisfiable.

$T_{\mathbb{Q}}$ Satisfiability II

- To solve F', consider

$$\begin{array}{ll}\textbf{max} & x_{n+1} \\ \textbf{subject to} & \\ & \bigwedge_{i=1}^{m} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i \\ & \bigwedge_{j=1}^{\ell} a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n + x_{n+1} \leq \beta_j \end{array}$$

- F' is $T_{\mathbb{Q}}$-satisfiable iff the optimum is positive.
- If F does not have any strict inequality, consider

$$\begin{array}{ll}\textbf{max} & 1 \\ \textbf{subject to} & \\ & \bigwedge_{i=1}^{m} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i \end{array}$$

- F is $T_{\mathbb{Q}}$-satisfiable iff the optimum is 1 by convention.

The Simplex Method

- Consider the generic linear program:

$$
\begin{array}{rll}
M: & \textbf{max} & \bar{c}^T\bar{x} \\
& \textbf{subject to} & \\
& & G : A\bar{x} \leq \bar{b}
\end{array}
$$

- The simplex method solves the linear program M by
  1. find an initial vertex $\bar{v}_1$ with $A\bar{v}_1 \leq \bar{b}$.
  2. at iteration i,
     1. if $\bar{v}_i$ maximizes the objective function among its adjacent vertices, return $\bar{v}_i$.
     2. otherwise, set $\bar{v}_{i+1}$ to an adjacent vertex with a greater objective value.
- Note that the simplex method returns a vertex $\bar{v}^*$ with a *local* optimum.
- By the convexity of polyhedra, $\bar{v}^*$ also attains the *global* optimum. (why?)

Finding Initial Vertex I

- Given $G : A\bar{x} \leq \bar{b}$, we construct another linear program M_0 to find an initial vertex in G.
- We first reformulate G as follows.
  - Replace a variable x with $x_+ - x_-$ where x_+ and x_- are non-negative.
  - Divide $A\bar{x} \leq \bar{b}$ into two by the signs of b_i to obtain

$$D_1\bar{x} \leq \bar{g}_1 \text{ and } D_2\bar{x} \geq \bar{g}_2 \text{ where } \bar{g}_1 \geq \bar{0}, \bar{g}_2 \geq \bar{0}.$$

- Hence we assume G has following form:

$$G : \begin{array}{rcl} \bar{x} & \geq & \bar{0} \\ D_1\bar{x} & \leq & \bar{g}_1 \\ D_2\bar{x} & \geq & \bar{g}_2 \end{array}$$

Finding Initial Vertex II

- Define

$$
M_0: \quad \textbf{max } \bar{1}^T(D_2\bar{x} - \bar{z}) \\
\textbf{subject to} \\
\begin{array}{rcl}
\bar{x} & \geq \bar{0} & \\
\bar{z} & \geq \bar{0} & \\
D_1\bar{x} & \leq & \bar{g}_1 \\
D_2\bar{x} - \bar{z} & \leq & \bar{g}_2
\end{array}
$$

- Observe that $\bar{x} = \bar{0}, \bar{z} = \bar{0}$ is a vertex in M_0.
  - $D_1\bar{0} = \bar{0} \leq \bar{g}_1$ and $D_2\bar{0} - \bar{0} = \bar{0} \leq \bar{g}_2$.
- Moreove, M_0 has the optimum $\bar{1}^T\bar{g}_2$ iff G is $T_{\mathbb{Q}}$-satisfiable.
  - Suppose the optimum $\bar{1}^T\bar{g}_2$ is attained by $\bar{x}^*, \bar{z}^*$. Then $D_2\bar{x}^* - \bar{z}^* = \bar{g}_2$. Thus $D_2\bar{x}^* = \bar{g}_2 + \bar{z}^* \geq \bar{g}_2$. We have $D_1\bar{x}^* \leq \bar{g}_1$ and $D_2\bar{x}^* \geq \bar{g}_2$. G is satisfied by $\bar{x}^*$.
  - Conversely, suppose $\bar{x}^*$ satisfies G. Then $D_1\bar{x}^* \leq \bar{g}_1$, $D_2\bar{x}^* \geq \bar{g}_2$, and $\bar{x}^* \geq \bar{0}$. Take $\bar{z}^* = D_2\bar{x}^* - \bar{g}_2$. Then $\bar{z}^* \geq \bar{0}$ and $D_2\bar{x}^* - \bar{z}^* \leq \bar{g}_2$. Furthermore, $\bar{1}^T(D_2\bar{x}^* - \bar{z}^*) = \bar{1}^T\bar{g}_2$ is the optimum.

Finding Initial Vertex III

- We can rearrange M_0 as a linear program.

$$M_0: \quad \textbf{max} \quad \bar{1}^T [D_2 - I] \begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix}$$

$$\textbf{subject to} \quad \begin{bmatrix} -I & \\ & -I \\ D_2 & \\ D_2 & -I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix} \leq \begin{bmatrix} \bar{0} \\ \bar{0} \\ \bar{g}_1 \\ \bar{g}_2 \end{bmatrix}$$

- $\bar{0}$ is an initial vertex in M_0.
- We just need to traverse adjacent vertices in M_0 to find the optimum of M_0.
- If the optimum is $\bar{1}^T \bar{g}_2$, we also find an initial vertex $\bar{v}_1$ in M.
  - Then we traverse adjacent vertices in M to find the optimum of M.
- Otherwise, G is not satisfiable. M has optimum $-\infty$.

Example I

- Consider $F : x + y \geq 1 \land x - y \geq -1$.
- F is equivalent to the following constraints:

$$G : \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Reformulate G and obtain

$$G : \begin{bmatrix} x_+ \\ x_- \\ y_+ \\ y_- \end{bmatrix} \geq \bar{0}, \begin{bmatrix} -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \\ y_+ \\ y_- \end{bmatrix} \leq [1]$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \\ y_+ \\ y_- \end{bmatrix} \geq [1]$$

Example II

- Hence the linear program M_0 is:

$$
\begin{array}{rl}
M_0: & \textbf{max} \quad \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_+ & x_- & y_+ & y_- \end{bmatrix}^T - [z] \\
& \textbf{subject to} \\
& \begin{array}{rcl}
\begin{bmatrix} x_+ & x_- & y_+ & y_- & z \end{bmatrix}^T & \geq & \bar{0} \\
\begin{bmatrix} -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \\ y_+ \\ y_- \end{bmatrix} & \leq & [1] \\
\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \\ y_+ \\ y_- \end{bmatrix} - [z] & \leq & [1]
\end{array}
\end{array}
$$

- F is $T_{\mathbb{Q}}$-satisfiable iff M_0 has optimum $\bar{1}^T \bar{g}_2 = 1$.

Example III

- M_0 can be rearranged in the generic form:

$$M_0: \quad \textbf{max} \quad \begin{bmatrix} 1 & -1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \\ y_+ \\ y_- \\ z \end{bmatrix}$$

$$\textbf{subject to} \quad \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \\ y_+ \\ y_- \\ z \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Vertex Traversal

- Assume that we have a vertex $\bar{v}_i$ in the following linear program:

$$
\begin{array}{lll}
M: & \textbf{max} & \bar{c}^T\bar{x} \\
& \textbf{subject to} & \\
& & A\bar{x} \leq \bar{b}
\end{array}
$$

- We find the vertex $\bar{v}^*$ maximizing $\bar{c}^T\bar{x}$ iteratively.
- At iteration i
  - A vertex $\bar{v}_i$ in M is known.
  - Check if $\bar{v}_i$ attains the maximum.
    - If the maximum is found, return $\bar{v}_i$.
    - Otherwise,find an adjacent vertex $\bar{v}_{i+1}$ with greater objective value.

Checking Maximum

- Let $\bar{v}_i$ be a vertex of $A\bar{x} \leq \bar{b}$.
- Hence there is an $n \times n$-submatrix A_i of A such that $A_i\bar{v}_i = \bar{b}_i$.
  - Recall that $A_i\bar{v}_i = \bar{b}_i$ is the defining constraints of $\bar{v}_i$.
  - There can be several such A_i. If so, choose a non-singular submatrix.
- Using high school mathematics, find $\bar{u}_i$ such that $A_i\bar{u}_i = \bar{c}_i$.
- Define $\bar{u}$ by extending missing dimensions of $\bar{u}_i$ with 0's.
- Hence we obtain an m-vector $\bar{u}$ with $\bar{u}^T A = \bar{c}^T$.

$\bar{u} \geq \bar{0}$

- If $\bar{u} \geq \bar{0}$, $\bar{v}_i$ indeed attains the maximum.
- This can be shown by applying Duality Theorem.
- Recall that $\bar{u}^T A = \bar{c}^T$ and $A_i \bar{v}_i = \bar{b}_i$.
- Let R be the row indices of A_i in A.
- Consider every row in $A\bar{v}_i$ and $\bar{b}$.
  - For $j \in R$, $A_i \bar{v}_i = \bar{b}_i$ and the j-term of $\bar{u}^T A \bar{v}_i =$ the j-term of $\bar{u}^T \bar{b}$.
  - For $j' \notin R$, $u_{j'} = 0$ and the j'-term of $\bar{u}^T A \bar{v}_i =$ the j'-term of $\bar{u}^T \bar{b}$.
- Hence we have $\bar{u}^T A \bar{v}_i = \bar{u}^T \bar{b}$.
- Since $\bar{u} \geq \bar{0}$ and $\bar{u}^T A = \bar{c}^T$, $\bar{u}^T \bar{b} \geq \min\{\bar{y}^T \bar{b} : \bar{y} \geq \bar{0}, \bar{y}^T A = \bar{c}^T\}$.
- By duality, $\min\{\bar{y}^T \bar{b} : \bar{y} \geq \bar{0}, \bar{y}^T A = \bar{c}^T\} = \max\{\bar{c}^T \bar{x} : A\bar{x} \leq \bar{b}\}$.
- Therefore,

$$\begin{aligned} \bar{c}^T \bar{v}_i = \bar{u}^T A \bar{v}_i = \bar{u}^T \bar{b} &\geq \min\{\bar{y}^T \bar{b} : \bar{y} \geq \bar{0}, \bar{y}^T A = \bar{c}^T\} \\ &= \max\{\bar{c}^T \bar{x} : A\bar{x} \leq \bar{b}\}. \end{aligned}$$

Example I

- Consider the vertex $\bar{v}_1 = [\ 0 \quad 0\ ]^T$ in

$$\begin{array}{ll} \textbf{max} & [\ -1 \quad 1\ ]\bar{x} \\ \textbf{subject to} & \\ & \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 2 & 1 \end{bmatrix} \bar{x} \leq \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \end{array}$$

- $\bar{u}_1 = [\ 1 \quad -1\ ]^T$ is a solution to

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^T \bar{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Hence $\bar{u} = [\ 1 \quad -1 \quad 0\ ]^T$.
- Note that $\bar{u} \not\geq \bar{0}$.

Example II

- Recall the example $F : x + y \geq 1 \land x - y \geq -1$.
- $\bar{v}_1 = [\ 0 \quad 0 \quad 0 \quad 0 \quad 0\ ]^T$ is a vertex with defining constraints

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \\ y_+ \\ y_- \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- $\bar{u}_1 = [\ -1 \quad 1 \quad -1 \quad 1 \quad 1\ ]^T$ is a solution to

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}^T \begin{bmatrix} x_+ \\ x_- \\ y_+ \\ y_- \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

- Hence $\bar{u} = [\ -1 \quad 1 \quad -1 \quad 1 \quad 1 \quad 0 \quad 0\ ]^T \not\geq \bar{0}$

$\bar{u} \ngeq \bar{0}$

- When $\bar{u} \ngeq \bar{0}$, $\bar{v}_i$ is not the optimal point.
- We want to find a direction $\bar{y}$ toward an adjacent vertex.
- If $\bar{u} \ngeq \bar{0}$, there must be a defining constraint corresponding to some $u_k < 0$.
- We move away from the corresponding defining constraint while preserving other defining constraints.

Finding the Direction $\bar{y}$

- Suppose $\bar{u} \not\geq \bar{0}$.
- Let k be the minimal index of $\bar{u}$ such that $u_k < 0$.
- Let k' be the index of $\bar{u}_i$ corresponding to k.
- Using high school mathematics, find $\bar{y}$ by solving

$$A_i \bar{y} = -e_{k'}.$$

- Observe
  - $\bar{a}_\ell \bar{y} = 0$ for every row $\bar{a}_\ell$ of A_i with index other than k'.
    - ★ Hence $\bar{y}$ preserves other defining constraints.
  - $\bar{a}_{k'} \bar{y} = -1$ for the k' row $\bar{a}_{k'}$ of A_i.
    - ★ Hence $\bar{y}$ moves away from the offending constraint.
- Consider $\bar{v}_i + \lambda \bar{y}$ for $\lambda > 0$.
  - $\bar{a}_\ell(\bar{v}_i + \lambda \bar{y}) = b_\ell$ for every row $\bar{a}_\ell$ of A_i with index other than k'.
- All defining constraints but k' are still satisfied!

Example I

- Recall $\bar{v}_1 = [\ 0 \quad 0\ ]^T$ and $\bar{u} = [\ 1 \quad -1 \quad 0\ ]^T$ in the following example:

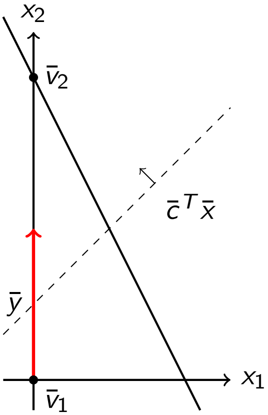
$$\begin{array}{l} \textbf{max} \quad [\ -1 \quad 1\ ]\bar{x} \\ \textbf{subject to} \\ \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 2 & 1 \end{bmatrix} \bar{x} \leq \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \end{array}$$

- The offending defining constraint is row $k' = 2$.
- Hence we find $\bar{y}$ by solving

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \bar{y} = -e_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

- $\bar{y} = [\ 0 \quad 1\ ]^T$.

Example II



$$\bar{v}_1 = [\begin{array}{cc} 0 & 0 \end{array}]^T \quad \bar{c} = [\begin{array}{cc} -1 & 1 \end{array}]^T \quad \bar{y} = [\begin{array}{cc} 0 & 1 \end{array}]^T$$

Example

- Recall $\bar{v}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ and $\bar{u}_1 = \begin{bmatrix} -1 & 1 & -1 & 1 & 1 \end{bmatrix}^T$ in

$$\begin{array}{ll} \textbf{max} & \begin{bmatrix} 1 & -1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_+ & x_- & y_+ & y_- & z \end{bmatrix}^T \\ \textbf{subject to} & \\ & \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \\ y_+ \\ y_- \\ z \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \end{array}$$

- An offending defining constraint is row $k' = 1$.
- We find $\bar{y}$ by solving $-I\bar{y} = -e_1$.
- $\bar{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$.

Obtaining $\bar{v}_{i+i}$ from $\bar{v}_i$ and $\bar{y}$

- Remember $\bar{y}$ is a "good" direction to move.
- Hence we try to find $\bar{v}_{i+1} = \bar{v}_i + \lambda_i \bar{y}$ with some $\lambda_i > 0$.
- There are two requirements for $\bar{v}_{i+1}$:
  - All constraints must still be satisfied.
  - $\bar{v}_{i+1}$ is a vertex.
- Therefore, we want to find the greatest $\lambda_i > 0$ such that $A(\bar{v}_i + \lambda_i \bar{y}) \leq \bar{b}$.
- When we find such a λ_i, set $\bar{v}_{i+1} = \bar{v}_i + \lambda_i \bar{y}$.
- There are two cases to consider:
  - $A\bar{y} \not\leq \bar{0}$, then λ_i exists.
  - $A\bar{y} \leq \bar{0}$, λ_i can be arbitrarily large (why?).

$A\bar{y} \not\leq \bar{0}$

- This is straightforward.
- Using high school mathematics to find another vertex.
- Consider the objective value

$$\begin{array}{rcl}
\bar{c}^T \bar{v}_{i+1} & = & \bar{c}^T(\bar{v}_i + \lambda_i \bar{y}) \\
 & = & \bar{c}^T \bar{v}_i + \lambda_i \bar{c}^T \bar{y} \\
 & = & \bar{c}^T \bar{v}_i + \lambda_i \bar{c}^T \bar{y} \\
 & = & \bar{c}^T \bar{v}_i + \lambda_i \bar{u}^T A \bar{y} \\
 & = & \bar{c}^T \bar{v}_i + \lambda_i \bar{u}^T (-e_k) \\
 & = & \bar{c}^T \bar{v}_i + \lambda_i (-u_k) \\
 & > & \bar{c}^T \bar{v}_i
\end{array}$$

- Moving from the vertex $\bar{v}_i$ to the vertex $\bar{v}_{i+1}$ always improves the objective value.

Example I

- Recall $\bar{v}_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $\bar{y} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ in

$$\begin{array}{ll} \textbf{max} & \begin{bmatrix} -1 & 1 \end{bmatrix} \bar{x} \\ \textbf{subject to} & \\ & \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 2 & 1 \end{bmatrix} \bar{x} \leq \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \end{array}$$

- Choose λ_1 such that

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \leq \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

- Recall that row 2 is the offending constraint.
- We now find a vertex defined by row 0 and 3.

Example II

- That is, solving

$$[\ 2 \quad 1\ ]([\ 0 \quad 0\ ]^T + \lambda_1[\ 0 \quad 1\ ]^T) = 2$$

- $\lambda_1 = 2$.
- Hence $\bar{v}_2 = \bar{v}_1 + \lambda_1 \bar{y} = [\ 0 \quad 0\ ]^T + 2[\ 0 \quad 1\ ]^T = [\ 0 \quad 2\ ]^T$.
- Recall that row 1 is satisfied for any λ_1.
- Since the defining constraints of $\bar{v}_2$ are row 0 and 3, we have

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \text{ and } \bar{b}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

- Let us find $\bar{u}_2$ by solving $A_2^T \bar{u}_2 = \bar{c}$:

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \bar{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- By high school mathematics, $\bar{u}_2 = \begin{bmatrix} 3 & 1 \end{bmatrix}^T$ and hence

$$\bar{u} = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}^T$$

- Since $\bar{u} \geq \bar{0}$, we have found the optimum

$$\bar{c}^T \bar{v}_2 = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2$$

$A\bar{y} \leq \bar{0}$

- In this case, the optimum is unbounded.
- Recall $A\bar{v}_i \leq \bar{b}$, we have

$$A(\bar{v}_i + \lambda\bar{y}) = A\bar{v}_i + \lambda A\bar{y} \leq \bar{b}.$$

- That is, $\bar{v}_i + \lambda\bar{y}$ satisfies all constraints for any $\lambda \geq 0$.
- Moreover, the objective value

$$\begin{aligned} \bar{c}^T(\bar{v}_i + \lambda\bar{y}) &= \bar{c}^T\bar{v}_i + \lambda\bar{c}^T\bar{y} \\ &= \bar{c}^T\bar{v}_i + \lambda\bar{u}^T A\bar{y} \\ &= \bar{c}^T\bar{v}_i + \lambda\bar{u}^T(-e_k) \\ &= \bar{c}^T\bar{v}_i - \lambda u_k. \end{aligned}$$

- Since $u_k < 0$, the objective value can be arbitrarily large as $\lambda \to \infty$.

Example I

- Recall $\bar{v}_1 = [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ]^T$ and $\bar{y} = [\ 1\ \ 0\ \ 0\ \ 0\ \ 0\ ]^T$ in
$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \\ y_+ \\ y_- \\ z \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$
- We try to find $\bar{v}_2$ defined by row 7 since
$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Example II

- Hence we find $\lambda_1 = 1$ by solving

$$[\begin{array}{ccccc} 1 & -1 & 1 & -1 & -1 \end{array}](\bar{v}_1^T + \lambda_1 [\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \end{array}]^T) = 1$$

- Thus $\bar{v}_2 = \bar{v}_1 + \lambda_1 \bar{y} = [\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \end{array}]^T$.
- The defining constraints for $\bar{v}_1$ are rows $7, 2, 3, 4, 5$.
- Hence

$$A_2 = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } \bar{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Example III

- Using high school mathematics, we now find $\bar{u}_2$ by solving $A_2^T \bar{u}_2 = \bar{c}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{bmatrix} \bar{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

- We have $\bar{u}_2 = [\ 1 \quad 0 \quad 0 \quad 0 \quad 0\ ]^T$.
- Since $\bar{u}_2 \geq \bar{0}$ and hence $\bar{u} \geq \bar{0}$, the optimum is

$$\bar{c}^T \bar{v}_2 = [\ 1 \quad -1 \quad 1 \quad -1 \quad -1\ ][\ 1 \quad 0 \quad 0 \quad 0 \quad 0\ ]^T = 1.$$

Complete Example

- Consider the $T_{\mathbb{Q}}$-formula:

$$F : x \geq 0 \land y \geq 0 \land x \geq 2 \land y \geq 2 \land x + y \leq 3.$$

- Equivalently, consider the linear program:

$$M : \quad \textbf{max} \quad 1$$
$$\textbf{subject to}$$
$$G : \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ -2 \\ -2 \\ 3 \end{bmatrix}$$

Complete Example – Constructing M_0

- Since G has constraints $x \geq 0$ and $y \geq 0$, we do not need x_+, x_-, y_+, y_-.
- Moreover,

$$\overbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}^{D_1} \begin{bmatrix} x \\ y \end{bmatrix} \leq \overbrace{\begin{bmatrix} 3 \end{bmatrix}}^{\bar{g}_1} \text{ and } \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{D_2} \begin{bmatrix} x \\ y \end{bmatrix} \geq \overbrace{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}^{\bar{g}_2}$$

- Hence

$$M_0 : \quad \textbf{max} \quad \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x & y & z_1 & z_2 \end{bmatrix}^T$$

$$\textbf{subject to}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

Iteration $i = 1$ I

- $\bar{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $A_1 = \begin{array}{cc} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$, $\bar{b}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.
- Obtain $\bar{u}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ by solving $A_1^T \bar{u}_1 = \bar{c} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.
- Row 1 is an offending constraint.
- Obtain $\bar{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ by solving $A_1 \bar{y} = -e_1$.

Iteration $i = 1$ II

- $A\bar{y} = [\ -1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0\ ]^T$.
- Find the maximal $\lambda = 2$ so that $\bar{v}_1 + \lambda\bar{y}$ satisfies rows 5 and 6.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \leq \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

- Hence $\bar{v}_2 = \bar{v}_1 + \lambda\bar{y} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

- $\bar{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $A_2 = \begin{array}{cc} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} & \begin{array}{c} 6 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$, $\bar{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.
- Obtain $\bar{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ by solving $A_2^T \bar{u}_2 = \bar{c} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.
- Row 2 is an offending constraint.
- Obtain $\bar{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ by solving $A_2 \bar{y} = -e_2$.

Iteration $i = 2$ II

- $A\bar{y} = [\ 0 \quad -1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1\ ]^T$.
- Find the maximal $\lambda = 1$ so that $\bar{v}_2 + \lambda\bar{y}$ satisfies rows 5 and 7.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) \leq \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

- Hence $\bar{v}_3 = \bar{v}_2 + \lambda\bar{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Iteration $i = 3$ I

- $\bar{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $A_3 = \begin{array}{cc} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} & \begin{matrix} 6 \\ 5 \\ 3 \\ 4 \end{matrix} \end{array}$, $\bar{b}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$.
- Obtain $\bar{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ by solving $A_3^T \bar{u}_3 = \bar{c} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.
- $\bar{u}_3 \geq \bar{0}$, $\bar{v}_3$ attains the optimum $\bar{c}^T \bar{v}_3 = 3$ of M_0.
- Recall that G is $T_{\mathbb{Q}}$-satisfiable iff the optimum of M_0 is $v_G = \bar{1}^T \bar{g}_2 = 4$.
- Hence the formula F is not $T_{\mathbb{Q}}$-satisfiable.