FLOLAC'12

Logic

Lecture 2: classical logic

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Classical semantics of propositional logic

Classical semantics adopts the *principle of bivalence*: every proposition denotes exactly one of the two truth-values, 0 (false) or 1 (true).

Definition. The set of *valuations* is defined to be $\mathcal{PV} \to \mathbf{2}$, where $\mathbf{2} := \{0, 1\}$.

Definition. Let σ be a valuation. The *truth-value interpretation* $[\![\_]\!]_\sigma : \textsc{Prop} \to \mathbf{2}$ of propositional formulas is defined by

$$
\begin{array}{lcll}
[\![\bot]\!]_\sigma & = & 0 & \\
[\![v]\!]_\sigma & = & \sigma\, v & \text{for } v : \mathcal{PV} \\
[\![\varphi \wedge \psi]\!]_\sigma & = & \mathit{min}\ [\![\varphi]\!]_\sigma\ [\![\psi]\!]_\sigma & \\
[\![\varphi \vee \psi]\!]_\sigma & = & \mathit{max}\ [\![\varphi]\!]_\sigma\ [\![\psi]\!]_\sigma & \\
[\![\varphi \to \psi]\!]_\sigma & = & \mathbf{if}\ [\![\varphi]\!]_\sigma \leq [\![\psi]\!]_\sigma\ \mathbf{then}\ 1\ \mathbf{else}\ 0 &
\end{array}
$$

Meta-connectives

Let σ be a valuation.

- $[\![\neg\varphi]\!]_\sigma = 1 \;\Leftrightarrow\; [\![\varphi]\!]_\sigma \leq [\![\bot]\!]_\sigma \;\Leftrightarrow\; [\![\varphi]\!]_\sigma \leq 0 \;\Leftrightarrow\; [\![\varphi]\!]_\sigma = 0.$
- $[\![\top]\!]_\sigma \;=\; [\![\neg\bot]\!]_\sigma \;=\; 1 - [\![\bot]\!]_\sigma \;=\; 1 - 0 \;=\; 1.$
- $$\begin{array}{ll} & [\![\varphi \leftrightarrow \psi]\!]_\sigma = 1 \\ \Leftrightarrow & \quad \{\text{ case `}\wedge\text{' }\} \\ & \mathit{min}\ [\![\varphi \rightarrow \psi]\!]_\sigma\ [\![\psi \rightarrow \varphi]\!]_\sigma = 1 \\ \Leftrightarrow & \quad \{\text{ arithmetic }\} \\ & [\![\varphi \rightarrow \psi]\!]_\sigma = 1 \quad \text{and} \quad [\![\psi \rightarrow \varphi]\!]_\sigma = 1 \\ \Leftrightarrow & \quad \{\text{ case `}\rightarrow\text{' }\} \\ & [\![\varphi]\!]_\sigma \leq [\![\psi]\!]_\sigma \quad \text{and} \quad [\![\psi]\!]_\sigma \leq [\![\varphi]\!]_\sigma \\ \Leftrightarrow & \quad \{\text{ antisymmetry }\} \\ & [\![\varphi]\!]_\sigma = [\![\psi]\!]_\sigma. \end{array}$$

Semantic definitions

Definitions. Let φ, ψ : $\textsc{Prop}$ and Γ : $\textsc{List Prop}$.

- A valuation σ *satisfies* φ if $[\![\varphi]\!]_\sigma = 1$; it satisfies Γ if it satisfies every formula in Γ.
- φ is a *semantic consequence* of Γ if, for any valuation σ, φ is satisfied by σ whenever Γ is satisfied by σ. In this case we write $\Gamma \models \varphi$.
- φ is *valid* if $\emptyset \models \varphi$. In this case φ is also called a *tautology*, and we simply write $\models \varphi$.
- φ and ψ are *semantically equivalent* if $[\![\varphi]\!]_\sigma = [\![\psi]\!]_\sigma$ for every valuation σ. In this case we write $\varphi \approx \psi$.

Example: $\models \varphi \vee \neg\varphi$

ASSUME $\sigma : \mathcal{PV} \to \mathbf{2}$

PROVE $[\![\varphi \vee \neg\varphi]\!]_\sigma = 1$

PROOF Case analysis on $[\![\varphi]\!]_\sigma$.

1 CASE $[\![\varphi]\!]_\sigma = 1$

PROOF $[\![\varphi \vee \neg\varphi]\!]_\sigma = \mathit{max}\ [\![\varphi]\!]_\sigma\ (1 - [\![\varphi]\!]_\sigma) = \mathit{max}\ 1\ 0 = 1.$

2 CASE $[\![\varphi]\!]_\sigma = 0$

PROOF $[\![\varphi \vee \neg\varphi]\!]_\sigma = \mathit{max}\ [\![\varphi]\!]_\sigma\ (1 - [\![\varphi]\!]_\sigma) = \mathit{max}\ 0\ 1 = 1.$

3 QED.

PROOF Either $[\![\varphi]\!]_\sigma = 1$ or $[\![\varphi]\!]_\sigma = 0$; 1 and 2.

Notation. "CASE C" abbreviates "ASSUME C PROVE QED".

$\models \varphi \vee \neg\varphi$ — truth table method

We may just summarise the case analysis on $[\![\varphi]\!]_\sigma$ and evaluation of the value of the entire propositional formula in a *truth table*.

φ	φ	$\vee$	$\neg$	φ
0	0	1	1	0
1	1	1	0	1

Theorem. Validity in classical propositional logic is *decidable*, i.e., there is a mechanical procedure that, given a propositional formula, decides whether it is valid or not in a finite amount of time.

Reducing connectives

We do not actually need that many connectives for classical propositional logic.

Definition. The set $\textsc{Prop}^-$ is inductively defined by the following rules:

- $\bot : \textsc{Prop}^-$;
- $v : \textsc{Prop}^-$ if $v : \mathcal{PV}$;
- $\varphi \to \psi : \textsc{Prop}^-$ if $\varphi, \psi : \textsc{Prop}^-$.

Definition. Let the function $\_^+ : \textsc{Prop}^- \to \textsc{Prop}$ be defined by

$$\begin{array}{lcll} \bot^+ & = & \bot & \\ v^+ & = & v & \text{for } v : \mathcal{PV} \\ (\varphi \to \psi)^+ & = & \varphi^+ \to \psi^+. & \end{array}$$

Reducing connectives

Theorem. For every $\varphi : \textsc{Prop}$, there exists $\varphi^- : \textsc{Prop}^-$ such that $\varphi \approx (\varphi^-)^+$.

PROOF Induction on φ.

1. $\perp \approx (\varphi^-)^+$ for some $\varphi^- : \textsc{Prop}^-$.

   PROOF Choose $\varphi^- := \perp$.

2. ASSUME $v : \mathcal{PV}$

   PROVE $v \approx (\varphi^-)^+$ for some $\varphi^- : \textsc{Prop}^-$

   PROOF Choose $\varphi^- := v$.

Reducing connectives

3 ASSUME $\psi : \textsc{Prop}$, $\psi^- : \textsc{Prop}^-$, $\psi \approx (\psi^-)^+$,
$\vartheta : \textsc{Prop}$, $\vartheta^- : \textsc{Prop}^-$, $\vartheta \approx (\vartheta^-)^+$

PROVE $\psi \wedge \vartheta \approx (\varphi^-)^+$ for some $\varphi^- : \textsc{Prop}^-$

PROOF Choose $\varphi^- := \neg(\psi^- \to \neg\vartheta^-)$, which is justified by the following truth table:

ψ	ϑ	ψ	$\wedge$	ϑ	$\leftrightarrow$	$\neg($	$(\psi^-)^+$	$\to$	$\neg$	$(\vartheta^-)^+)$
0	0	0	0	0	1	0	0	1	1	0
0	1	0	0	1	1	0	0	1	0	1
1	0	1	0	0	1	0	1	1	1	0
1	1	1	1	1	1	1	1	0	0	1

Lemma. $\varphi \approx \psi$ if and only if $\models \varphi \leftrightarrow \psi$.

Reducing connectives

4 ASSUME $\psi : \textsc{Prop},\ \ \psi^- : \textsc{Prop}^-,\ \ \psi \approx (\psi^-)^+,$
$\vartheta : \textsc{Prop},\ \ \vartheta^- : \textsc{Prop}^-,\ \ \vartheta \approx (\vartheta^-)^+$

PROVE $\psi \vee \vartheta \approx (\varphi^-)^+$ for some $\varphi^- : \textsc{Prop}^-$

PROOF Choose $\varphi^- := \neg\psi^- \to \vartheta^-$, which is justified by the following truth table:

ψ	ϑ	ψ	$\vee$	ϑ	$\leftrightarrow$	$(\neg$	$(\psi^-)^+$	$\to$	$(\vartheta^-)^+)$
0	0	0	0	0	1	1	0	0	0
0	1	0	1	1	1	1	0	1	1
1	0	1	1	0	1	0	1	1	0
1	1	1	1	1	1	0	1	1	1

Reducing connectives

5 ASSUME $\psi : \textsc{Prop},\ \psi^- : \textsc{Prop}^-,\ \psi \approx (\psi^-)^+,$
$\vartheta : \textsc{Prop},\ \vartheta^- : \textsc{Prop}^-,\ \vartheta \approx (\vartheta^-)^+$

PROVE $\psi \to \vartheta \approx (\varphi^-)^+$ for some $\varphi^- : \textsc{Prop}^-$

PROOF Choose $\varphi^- := \psi^- \to \vartheta^-$.

6 QED.

PROOF By $\boxed{1}$–$\boxed{5}$ and the induction principle on $\textsc{Prop}$.

Soundness

From now on we assume that $\_^{+}$ is implicitly applied where needed.

Theorem. $\vdash_{\mathrm{NJ}^-} \varphi$ implies $\models \varphi$ for every $\varphi : \textsc{Prop}^-$.

PROOF Semantic truth is preserved by every deduction rule.

1 $\Gamma \vdash_{\mathrm{NJ}^-} \varphi$ implies $\Gamma \models \varphi$ for every $\varphi : \textsc{Prop}^-$ and $\Gamma : \textsc{List Prop}^-$.

PROOF Induction on the derivation of $\Gamma \vdash_{\mathrm{NJ}^-} \varphi$.

2 QED.

PROOF Choose $\Gamma := \emptyset$ in $\boxed{1}$.

Inductive definition of derivations

Definition. The sets $\mathrm{NJ}^{-}[\Gamma;\varphi]$ of (closed) derivations, where Γ ranges over $\textsc{List Prop}^{-}$ and φ over $\textsc{Prop}^{-}$, are inductively defined by the following rules:

- $\dfrac{}{\Gamma \vdash \varphi}\,(\text{assum}) : \mathrm{NJ}^{-}[\Gamma;\varphi]$ if $\varphi \in \Gamma$;
- $\dfrac{d}{\Gamma \vdash \varphi}\,(\bot\text{E}) : \mathrm{NJ}^{-}[\Gamma;\varphi]$ if $d : \mathrm{NJ}^{-}[\Gamma;\bot]$;
- $\dfrac{d}{\Gamma \vdash \varphi \to \psi}\,(\to\text{I}) : \mathrm{NJ}^{-}[\Gamma;\varphi \to \psi]$ if $d : \mathrm{NJ}^{-}[\Gamma,\varphi;\psi]$;
- $\dfrac{d \qquad e}{\Gamma \vdash \psi}\,(\to\text{E}) : \mathrm{NJ}^{-}[\Gamma;\psi]$ if $d : \mathrm{NJ}^{-}[\Gamma;\varphi \to \psi]$ and
  $e : \mathrm{NJ}^{-}[\Gamma;\varphi]$.

Definition. φ is *derivable* from Γ in NJ^{-} if the set $\mathrm{NJ}^{-}[\Gamma;\varphi]$ is inhabited. In this case we write $\Gamma \vdash_{\mathrm{NJ}^{-}} \varphi$.

Treating proofs as formal objects is the defining characteristic of *proof theory*.

Induction principle on NJ^-

The rule

- $\dfrac{d}{\Gamma \vdash \varphi \to \psi}$ ($\to$I) : $\mathrm{NJ}^-[\Gamma; \varphi \to \psi]$ if $d : \mathrm{NJ}^-[\Gamma, \varphi; \psi]$

is interpreted as "if d is a derivation with conclusion $\Gamma, \varphi \vdash \psi$, then $\dfrac{d}{\Gamma \vdash \varphi \to \psi}$ ($\to$I) is a derivation with conclusion $\Gamma \vdash \varphi \to \psi$".

Let $P\ \Gamma\ \varphi\ d$ be a property on $\Gamma : \textsc{List Prop}^-$, $\varphi : \textsc{Prop}^-$, and $d : \mathrm{NJ}^-[\Gamma; \varphi]$, i.e., P talks about a derivation d and the context Γ and formula φ in the conclusion of d. The corresponding case of the above rule in the induction principle on NJ^- is

- For any $\Gamma : \textsc{List Prop}^-$, φ, $\psi \in \textsc{Prop}^-$, and $d : \mathrm{NJ}^-[\Gamma, \varphi; \psi]$, $P\ \Gamma\ (\varphi \to \psi)\ \left(\dfrac{d}{\Gamma \vdash \varphi \to \psi}\,(\to\mathrm{I})\right)$ holds if $P\ (\Gamma, \varphi)\ \psi\ d$ holds.

Soundness

1 $\Gamma \vdash_{\mathrm{NJ}^-} \varphi$ implies $\Gamma \models \varphi$ for every $\varphi : \textsc{Prop}^-$ and $\Gamma : \textsc{List Prop}^-$.

PROVE $\Gamma \models \varphi$ holds for every $\Gamma : \textsc{List Prop}^-$ and $\varphi : \textsc{Prop}^-$ such that $\Gamma \vdash_{\mathrm{NJ}^-} \varphi$.

PROOF Induction on the derivation of $\Gamma \vdash_{\mathrm{NJ}^-} \varphi$.

1.1 Case (assum).

ASSUME $\Gamma : \textsc{List Prop}^-$, $\varphi : \textsc{Prop}^-$, $\varphi \in \Gamma$, $\sigma : \mathcal{PV} \to \mathbf{2}$, σ satisfies Γ

PROVE $[\![\varphi]\!]_\sigma = 1$

PROOF Since σ satisfies Γ and $\varphi \in \Gamma$.

Soundness

1.2 Case ($\bot$E).

ASSUME $\Gamma : \textsc{List Prop}^-$, $\Gamma \vdash_{\mathrm{NJ}^-} \bot$, $\Gamma \models \bot$,
$\varphi : \textsc{Prop}^-$, $\sigma : \mathcal{PV} \to \mathbf{2}$, σ satisfies Γ

PROVE $[\![\varphi]\!]_\sigma = 1$

PROOF Such σ could not have been given.

1.2.1 $[\![\bot]\!]_\sigma = 0$.

PROOF By definition.

1.2.2 $[\![\bot]\!]_\sigma = 1$.

PROOF $\Gamma \models \bot$ and σ satisfies Γ.

1.2.3 QED.

PROOF 1.2.1 and 1.2.2 are contradictory —
invoke the principle of explosion.

Soundness

1.3 Case ($\to$I).

ASSUME $\Gamma : \textsc{List Prop}^-$, $\varphi, \psi : \textsc{Prop}^-$,
$\Gamma, \varphi \vdash_{\mathrm{NJ}^-} \psi$, $\Gamma, \varphi \models \psi$,
$\sigma : \mathcal{PV} \to \mathbf{2}$, σ satisfies Γ

PROVE $[\![\varphi \to \psi]\!]_\sigma = 1$

PROOF Case analysis on the truth value of φ.

1.3.1 CASE $[\![\varphi]\!]_\sigma = 0$

1.3.2 CASE $[\![\varphi]\!]_\sigma = 1$

1.3.3 QED.

PROOF By $\boxed{1.3.1}$ and $\boxed{1.3.2}$.

Soundness

1.3.1 CASE $[\![\varphi]\!]_\sigma = 0$

PROOF

$$
\begin{array}{ll}
 & [\![\varphi \to \psi]\!]_\sigma = 1 \\
\Leftrightarrow & \quad \{\text{ definition of truth }\} \\
 & [\![\varphi]\!]_\sigma \leq [\![\psi]\!]_\sigma \\
\Leftrightarrow & \quad \{\text{ assumption }\} \\
 & 0 \leq [\![\psi]\!]_\sigma \\
\Leftrightarrow & \quad \{\text{ truth value is either } 0 \text{ or } 1 \text{ }\} \\
 & \textit{true}.
\end{array}
$$

Soundness

1.3.2 CASE $[\![\varphi]\!]_\sigma = 1$

PROOF ψ must be true, and therefore so must $\varphi \to \psi$.

1.3.2.1 σ satisfies Γ, φ.

PROOF σ satisfies Γ and φ.

1.3.2.2 $[\![\psi]\!]_\sigma = 1$.

PROOF $\Gamma, \varphi \models \psi$ and 1.3.2.1.

1.3.2.3 QED.

PROOF

$$
\begin{array}{ll}
 & [\![\varphi \to \psi]\!]_\sigma = 1 \\
\Leftrightarrow & \quad \{\text{ definition of truth }\} \\
 & [\![\varphi]\!]_\sigma \leq [\![\psi]\!]_\sigma \\
\Leftrightarrow & \quad \{\text{ 1.3.2.2 }\} \\
 & [\![\varphi]\!]_\sigma \leq 1 \\
\Leftrightarrow & \quad \{\text{ truth value is either } 0 \text{ or } 1 \text{ }\} \\
 & \textit{true}.
\end{array}
$$

Soundness

1.4 Case ($\to$E).

ASSUME $\Gamma : \textsc{List Prop}^-$, $\varphi, \psi : \textsc{Prop}^-$,
$\Gamma \vdash_{\mathrm{NJ}^-} \varphi \to \psi$, $\Gamma \models \varphi \to \psi$, $\Gamma \vdash_{\mathrm{NJ}^-} \varphi$, $\Gamma \models \varphi$,
$\sigma : \mathcal{PV} \to \mathbf{2}$, σ satisfies Γ

PROVE $[\![\psi]\!]_\sigma = 1$

PROOF Left as an exercise.

1.5 QED.

PROOF By 1.1–1.4 and the induction principle on NJ^-.

Semantic completeness

Non-theorem. $\models \varphi$ implies $\vdash_{\mathrm{NJ}^-} \varphi$ for any $\varphi : \textsc{Prop}^-$.

Counterexample. We have $\models \neg\neg\mathtt{A} \to \mathtt{A}$ but not $\vdash_{\mathrm{NJ}^-} \neg\neg\mathtt{A} \to \mathtt{A}$.

If, however, we extend NJ^- to NK^- with the rule

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}\ (\neg\neg\mathsf{E})$$

we do obtain semantic completeness of NK^- with respect to the truth-value semantics.

Theorem. $\models \varphi$ implies $\vdash_{\mathrm{NK}^-} \varphi$ for any $\varphi : \textsc{Prop}^-$.

The NK^- deduction system

Definition. The sets $\mathrm{NK}^-[\Gamma;\varphi]$ of (closed) derivations, where Γ ranges over $\textsc{List Prop}^-$ and φ over $\textsc{Prop}^-$, are inductively defined by the following rules:

- $\dfrac{}{\Gamma \vdash \varphi}\,(\text{assum}) : \mathrm{NK}^-[\Gamma;\varphi]$ if $\varphi \in \Gamma$;
- $\dfrac{d}{\Gamma \vdash \varphi}\,(\bot\mathsf{E}) : \mathrm{NK}^-[\Gamma;\varphi]$ if $d : \mathrm{NK}^-[\Gamma;\bot]$;
- $\dfrac{d}{\Gamma \vdash \varphi \to \psi}\,({\to}\mathsf{I}) : \mathrm{NK}^-[\Gamma;\varphi \to \psi]$ if $d : \mathrm{NK}^-[\Gamma,\varphi;\psi]$;
- $\dfrac{d \qquad e}{\Gamma \vdash \psi}\,({\to}\mathsf{E}) : \mathrm{NK}^-[\Gamma;\psi]$ if $d : \mathrm{NK}^-[\Gamma;\varphi \to \psi]$ and $e : \mathrm{NK}^-[\Gamma;\varphi]$;
- $\dfrac{d}{\Gamma \vdash \varphi}\,(\neg\neg\mathsf{E}) : \mathrm{NK}^-[\Gamma;\varphi]$ if $d : \mathrm{NK}^-[\Gamma;\neg\neg\varphi]$.

Definition. φ is *derivable* from Γ in NK^- if the set $\mathrm{NK}^-[\Gamma;\varphi]$ is inhabited. In this case we write $\Gamma \vdash_{\mathrm{NK}^-} \varphi$.

Weakening lemma

Lemma. Let $\Gamma, \Gamma' : \textsc{List Prop}^-$ such that $\Gamma \subseteq \Gamma'$ (i.e., $\varphi \in \Gamma$ implies $\varphi \in \Gamma'$ for any φ). Then $\Gamma \vdash_{\mathrm{NK}^-} \varphi$ implies $\Gamma' \vdash_{\mathrm{NK}^-} \varphi$ for any $\varphi : \textsc{Prop}^-$.

PROOF Induction on the derivation of $\Gamma \vdash_{\mathrm{NK}^-} \varphi$.

Reconstruction lemma

Definition. The function $\mathit{vars} : \textsc{Prop}^{-} \to \textsc{List}\ \mathcal{PV}$, which computes the list of propositional variables occurring in a propositional formula, is defined by

$$\begin{array}{lcll} \mathit{vars}\ \bot & = & \emptyset & \\ \mathit{vars}\ v & = & [v] & \text{for } v : \mathcal{PV} \\ \mathit{vars}\ (\varphi \to \psi) & = & \mathit{vars}\ \varphi \cup \mathit{vars}\ \psi & \end{array}$$

Lemma. Let $\varphi : \textsc{Prop}^{-}$ and $\sigma : \mathcal{PV} \to \mathbf{2}$. Define

$$T_\sigma\ \psi \ := \ \mathbf{if}\ [\![\psi]\!]_\sigma = 1\ \mathbf{then}\ \psi\ \mathbf{else}\ \neg\psi.$$

Then $T_\sigma\ (\mathit{vars}\ \varphi) \vdash_{\mathrm{NK}^{-}} T_\sigma\ \varphi$, where

$$T_\sigma\ (\mathit{vars}\ \varphi) = [\ T_\sigma\ v \mid v \in \mathit{vars}\ \varphi\ ].$$

$\boxed{\text{PROOF}}$ Induction on φ.

Reconstruction lemma

1. $\vdash_{\mathrm{NK}^-} \neg\bot$.

   PROOF $\dfrac{\overline{\bot \vdash \bot}}{\vdash \neg\bot}\ (\to\mathrm{I})$

2. ASSUME $v : \mathcal{PV}$

   PROVE $T_\sigma\, v \vdash_{\mathrm{NK}^-} T_\sigma\, v$

   PROOF $\dfrac{}{T_\sigma\, v \vdash T_\sigma\, v}\ (\mathrm{assum})$

Reconstruction lemma

3 ASSUME $\varphi : \mathrm{P{\scriptsize ROP}}^-$, $T_\sigma\ (\mathit{vars}\ \varphi) \vdash_{\mathrm{NK}^-} T_\sigma\ \varphi$,
$\psi : \mathrm{P{\scriptsize ROP}}^-$, $T_\sigma\ (\mathit{vars}\ \psi) \vdash_{\mathrm{NK}^-} T_\sigma\ \psi$

PROVE $T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)) \vdash_{\mathrm{NK}^-} T_\sigma\ (\varphi \to \psi)$

PROOF Case analysis to determine $T_\sigma\ (\varphi \to \psi)$.

3.1 CASE $[\![\psi]\!]_\sigma = 1$

3.2 CASE $[\![\psi]\!]_\sigma = 0$

PROOF Left as an exercise.

3.3 QED.

PROOF By $\boxed{3.1}$ and $\boxed{3.2}$.

4 QED.

PROOF By $\boxed{1}$–$\boxed{3}$ and the induction principle on $\mathrm{P{\scriptsize ROP}}^-$.

Reconstruction lemma

3.1 CASE $[\![\psi]\!]_\sigma = 1$

PROOF In this case $T_\sigma\ (\varphi \to \psi) = \varphi \to \psi$. We can prove ψ and thus $\varphi \to \psi$.

3.1.1 $T_\sigma\ (\mathit{vars}\ \psi) \vdash_{\mathrm{NK}^-} \psi$.

PROOF Induction hypothesis $T_\sigma\ (\mathit{vars}\ \psi) \vdash_{\mathrm{NK}^-} T_\sigma\ \psi$, where $T_\sigma\ \psi = \psi$ since $[\![\psi]\!]_\sigma = 1$.

3.1.2 LET $d : \mathrm{NK}^-[T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)), \varphi; \psi]$

PROOF 3.1.1 and weakening.

3.1.3 QED.

PROOF

$$\frac{d}{T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)) \vdash \varphi \to \psi}\ (\to\mathrm{I})$$

Semantic completeness

Theorem. $\models \varphi$ implies $\vdash_{\mathrm{NK}^-} \varphi$ for any $\varphi : \textsc{Prop}^-$.

PROOF Construct a derivation that encodes the truth table, where each sub-derivation that encodes an entry of the table is produced by the reconstruction lemma.

1 For any finite $\Gamma : \textsc{List}\ \mathcal{PV}$ whose elements are all distinct, $T_\sigma\ \Gamma \vdash_{\mathrm{NK}^-} \varphi$ for any σ implies $\vdash_{\mathrm{NK}^-} \varphi$.

PROOF Induction on (the size of) Γ.

2 QED.

PROOF Choose $\Gamma := \mathit{vars}\ \varphi$ in $\boxed{1}$.

Semantic completeness

2 QED.

PROOF Choose $\Gamma := \mathit{vars}\ \varphi$ in $\boxed{1}$.

2.1 $T_\sigma\ (\mathit{vars}\ \varphi) \vdash_{\mathrm{NK}^-} T_\sigma\ \varphi$ for any σ.

PROOF By the reconstruction lemma.

2.2 $T_\sigma\ (\mathit{vars}\ \varphi) \vdash_{\mathrm{NK}^-} \varphi$ for any σ.

PROOF In $\boxed{2.1}$, $T_\sigma\ \varphi = \varphi$ for any σ since φ is valid.

2.3 QED.

PROOF Choose $\Gamma := \mathit{vars}\ \varphi$ in $\boxed{1}$ and discharge the condition by $\boxed{2.2}$.

Semantic completeness

1 For any finite $\Gamma : \textsc{List}\ \mathcal{PV}$ whose elements are all distinct, $T_\sigma\ \Gamma \vdash_{\mathrm{NK}^-} \varphi$ for any σ implies $\vdash_{\mathrm{NK}^-} \varphi$.

PROOF Induction on (the size of) Γ.

1.1 $\vdash_{\mathrm{NK}^-} \varphi$ for any σ implies $\vdash_{\mathrm{NK}^-} \varphi$.

PROOF Use the condition by choosing an arbitrary σ.

1.2 ASSUME $\Gamma : \textsc{List}\ \mathcal{PV}$ consisting of a finite number of distinct elements, $v : \mathcal{PV}$, $v \notin \Gamma$,
$T_\sigma\ \Gamma \vdash_{\mathrm{NK}^-} \varphi$ for any σ implies $\vdash_{\mathrm{NK}^-} \varphi$,
$T_\sigma\ \Gamma, T_\sigma\ v \vdash_{\mathrm{NK}^-} \varphi$ for any σ

PROVE $\vdash_{\mathrm{NK}^-} \varphi$

1.3 QED.

PROOF By 1.1 – 1.2 and the induction principle on lists.

Semantic completeness

1.2 ASSUME $\Gamma : \textsc{List}\ \mathcal{PV}$ consisting of a finite number of distinct elements, $v : \mathcal{PV}$, $v \notin \Gamma$,
$T_\sigma\ \Gamma \vdash_{\mathrm{NK}^-} \varphi$ for any σ implies $\vdash_{\mathrm{NK}^-} \varphi$,
$T_\sigma\ \Gamma, T_\sigma\ v \vdash_{\mathrm{NK}^-} \varphi$ for any σ

PROVE $\vdash_{\mathrm{NK}^-} \varphi$

PROOF Use the induction hypothesis.

1.2.1 $T_\sigma\ \Gamma, v \vdash_{\mathrm{NK}^-} \varphi$ for any σ.

1.2.2 $T_\sigma\ \Gamma, \neg v \vdash_{\mathrm{NK}^-} \varphi$ for any σ.

1.2.3 $T_\sigma\ \Gamma \vdash_{\mathrm{NK}^-} \varphi$ for any σ.

1.2.4 QED.

PROOF Discharge the condition of the induction hypothesis by 1.2.3.

Semantic completeness

1.2.1 $T_\sigma\ \Gamma, v \vdash_{\mathrm{NK}^-} \varphi$ for any σ.

PROOF Given σ, instantiate the last assumption with $\sigma\,[1/v]$. Then $T_{\sigma\,[1/v]}\ v = v$, and $T_{\sigma\,[1/v]}\ \Gamma = T_\sigma\ \Gamma$ since $v \notin \Gamma$.

1.2.2 $T_\sigma\ \Gamma, \neg v \vdash_{\mathrm{NK}^-} \varphi$ for any σ.

PROOF Similar to 1.2.1 but instantiating the assumption with $\sigma\,[0/v]$.

Definition. Let $f\colon S \to T$, $s : S$, and $t : T$. The function $f\,[t/s] : S \to T$ is defined by

$$(f\,[t/s])\ x \ := \ \mathbf{if}\ x = s\ \mathbf{then}\ t\ \mathbf{else}\ f\,x.$$

Semantic completeness

1.2.3 $T_\sigma\ \Gamma \vdash_{\mathrm{NK}^-} \varphi$ for any σ.

ASSUME $\sigma : \mathcal{PV} \to \mathbf{2}$

PROVE $T_\sigma\ \Gamma \vdash_{\mathrm{NK}^-} \varphi$

PROOF Refute $\neg\varphi$ and use double negation elimination.

1.2.3.1 LET $d : \mathrm{NK}^-[T_\sigma\ \Gamma, \neg\varphi, v;\ \varphi]$

PROOF 1.2.1 and weakening.

1.2.3.2 LET $e : \mathrm{NK}^-[T_\sigma\ \Gamma, \neg\varphi, \neg v;\ \varphi]$

PROOF 1.2.2 and weakening.

1.2.3.3 QED.

Semantic completeness

1.2.3.3 QED.

PROOF

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta, \neg v \vdash \neg\varphi} \qquad e}{\Delta, \neg v \vdash \bot}\,(\to\mathrm{E})}{\Delta \vdash \neg\neg v}\,(\to\mathrm{I}) \qquad \dfrac{\dfrac{\dfrac{}{\Delta, v \vdash \neg\varphi} \qquad d}{\Delta, v \vdash \bot}\,(\to\mathrm{E})}{\Delta \vdash \neg v}\,(\to\mathrm{I})}{\overbrace{T_\sigma\ \Gamma, \neg\varphi}^{\Delta} \vdash \bot}\,(\to\mathrm{E})}{T_\sigma\ \Gamma \vdash \neg\neg\varphi}\,(\to\mathrm{I})}{T_\sigma\ \Gamma \vdash \varphi}\,(\neg\neg\mathrm{E})
$$

Classical semantics of first-order logic

Definition. Given a signature $\mathcal{S} = (\mathcal{P}, \mathcal{F})$, an *$\mathcal{S}$-structure* $\mathcal{M}$ consists of

- a nonempty set called the *domain*, which is simply denoted by $\mathcal{M}$,
- a function $[\![p]\!]_{\mathcal{M}} : (\mathcal{M} \to)^n\ \mathbf{2}$ for each predicate symbol $p : \mathcal{P}$ of arity n, and
- a function $[\![f]\!]_{\mathcal{M}} : (\mathcal{M} \to)^n\ \mathcal{M}$ for each function symbol $f : \mathcal{F}$ of arity n.

Definition. Given a structure $\mathcal{M}$, the set of *$\mathcal{M}$-assignments* is defined to be $\mathcal{IV} \to \mathcal{M}$.

Classical semantics of first-order logic

Definition. Let $\mathcal{S} = (\mathcal{P}, \mathcal{F})$ be a signature, $\mathcal{M}$ an $\mathcal{S}$-structure, and σ an $\mathcal{M}$-assignment. The *truth-value interpretation* $[\![\_]\!]_{\mathcal{M},\sigma} : \mathrm{F{\scriptsize ORM}}_{\mathcal{S}} \to \mathbf{2}$ of formulas is defined as follows:

$$
\begin{array}{lcll}
[\![\bot]\!]_{\mathcal{M},\sigma} & = & 0 & \\
[\![p\ t_1 \ldots t_n]\!]_{\mathcal{M},\sigma} & = & [\![p]\!]_{\mathcal{M}}\ \ [\![t_1]\!]_{\mathcal{M},\sigma} \ldots [\![t_n]\!]_{\mathcal{M},\sigma} & \text{for } p : \mathcal{P} \\
[\![\varphi \wedge \psi]\!]_{\mathcal{M},\sigma} & = & \mathit{min}\ \ [\![\varphi]\!]_{\mathcal{M},\sigma}\ \ [\![\psi]\!]_{\mathcal{M},\sigma} & \\
[\![\varphi \vee \psi]\!]_{\mathcal{M},\sigma} & = & \mathit{max}\ \ [\![\varphi]\!]_{\mathcal{M},\sigma}\ \ [\![\psi]\!]_{\mathcal{M},\sigma} & \\
[\![\varphi \to \psi]\!]_{\mathcal{M},\sigma} & = & \textbf{if } [\![\varphi]\!]_{\mathcal{M},\sigma} \leq [\![\psi]\!]_{\mathcal{M},\sigma} \textbf{ then } 1 \textbf{ else } 0 & \\
[\![\forall\, v.\ \varphi]\!]_{\mathcal{M},\sigma} & = & \textbf{if } [\![\varphi]\!]_{\mathcal{M},\sigma[m/v]} = 1 \text{ for every } m : \mathcal{M} & \\
 & & \qquad \textbf{then } 1 \textbf{ else } 0 & \\
[\![\exists\, v.\ \varphi]\!]_{\mathcal{M},\sigma} & = & \textbf{if } [\![\varphi]\!]_{\mathcal{M},\sigma[m/v]} = 0 \text{ for every } m : \mathcal{M} & \\
 & & \qquad \textbf{then } 0 \textbf{ else } 1 &
\end{array}
$$

where $[\![\_]\!]_{\mathcal{M},\sigma} : \mathrm{T{\scriptsize ERM}}_{\mathcal{F}} \to \mathcal{M}$ is defined as follows:

$$
\begin{array}{lcll}
[\![v]\!]_{\mathcal{M},\sigma} & = & \sigma\, v & \text{for } v : \mathcal{IV} \\
[\![f\ t_1 \ldots t_n]\!]_{\mathcal{M},\sigma} & = & [\![f]\!]_{\mathcal{M}}\ \ [\![t_1]\!]_{\mathcal{M},\sigma} \ldots [\![t_n]\!]_{\mathcal{M},\sigma} & \text{for } f : \mathcal{F}.
\end{array}
$$

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Semantic definitions

Definitions. Let $\mathcal{S}$ be a signature, φ, $\psi : \textsc{Form}_{\mathcal{S}}$, and $\Gamma : \textsc{List Form}_{\mathcal{S}}$.

- An $\mathcal{S}$-structure $\mathcal{M}$ and an $\mathcal{M}$-assignment σ *satisfy* φ if $[\![\varphi]\!]_{\mathcal{M},\sigma} = 1$; they satisfy Γ if they satisfy every formula in Γ.
- φ is a *semantic consequence* of Γ if, for any $\mathcal{S}$-structure $\mathcal{M}$ and $\mathcal{M}$-assignment σ, φ is satisfied by $\mathcal{M}$ and σ whenever Γ is satisfied by $\mathcal{M}$ and σ. In this case we write $\Gamma \models \varphi$.
- φ is *valid* if $\emptyset \models \varphi$. In this case we also call φ a *tautology* and simply write $\models \varphi$.
- φ and ψ are *semantically equivalent*, written $\varphi \approx \psi$, if $[\![\varphi]\!]_{\mathcal{M},\sigma} = [\![\psi]\!]_{\mathcal{M},\sigma}$ for every $\mathcal{S}$-structure $\mathcal{M}$ and $\mathcal{M}$-assignment σ.

Semantic definitions

Definitions. Let $\mathcal{M}$ be a structure, φ a sentence, and $\mathcal{T}$ a theory.

- $\mathcal{M}$ *satisfies* φ if φ is satisfied by $\mathcal{M}$ and any $\mathcal{M}$-assignment σ. In this case we call $\mathcal{M}$ a *model* of φ and write $\mathcal{M} \models \varphi$.
- $\mathcal{M}$ *satisfies* $\mathcal{T}$ if $\mathcal{M}$ satisfies every axiom in $\mathcal{T}$. In this case we call $\mathcal{M}$ a *model* of $\mathcal{T}$ and write $\mathcal{M} \models \mathcal{T}$.
- $\mathcal{T}$ is *satisfiable* or (semantically) *consistent* if it has a model.

This satisfaction relation is at the heart of *model theory*, which we do not cover in this course.

Example: $\models \neg(\forall\, v.\ \neg\varphi) \leftrightarrow \exists\, v.\ \varphi$

Equivalently we can prove that $\neg(\forall\, v.\ \neg\varphi) \approx \exists\, v.\ \varphi$.

ASSUME $\mathcal{S}$: signature, $\mathcal{M}$: $\mathcal{S}$-structure, σ : $\mathcal{M}$-assignment

PROVE $[\![\neg(\forall\, v.\ \neg\varphi)]\!]_{\mathcal{M},\sigma} = 1$ if and only if $[\![\exists\, v.\ \varphi]\!]_{\mathcal{M},\sigma} = 1$

Example: $\models \neg(\forall\, v.\ \neg\varphi) \leftrightarrow \exists\, v.\ \varphi$

PROOF By definition of truth-value interpretation.

$$
\begin{array}{ll}
 & [\![\neg(\forall\, v.\ \neg\varphi)]\!]_{\mathcal{M},\,\sigma} = 1 \\
\Leftrightarrow & \quad \{\text{ truth value of `}\neg\text{' }\} \\
 & [\![\forall\, v.\ \neg\varphi]\!]_{\mathcal{M},\,\sigma} = 0 \\
\Leftrightarrow & \quad \{\text{ truth value is either } 0 \text{ or } 1\ \} \\
 & \text{it is not the case that } [\![\forall\, v.\ \neg\varphi]\!]_{\mathcal{M},\,\sigma} = 1 \\
\Leftrightarrow & \quad \{\text{ truth value of `}\forall\text{' }\} \\
 & \text{it is not the case that } [\![\neg\varphi]\!]_{\mathcal{M},\,\sigma[m/v]} = 1 \text{ for every } m : \mathcal{M} \\
\Leftrightarrow & \quad \{\text{ truth value of `}\neg\text{' }\} \\
 & \text{it is not the case that } [\![\varphi]\!]_{\mathcal{M},\,\sigma[m/v]} = 0 \text{ for every } m : \mathcal{M} \\
\Leftrightarrow & \quad \{\text{ truth value of `}\exists\text{' }\} \\
 & \text{it is not the case that } [\![\exists\, v.\ \varphi]\!]_{\mathcal{M},\,\sigma} = 0 \\
\Leftrightarrow & \quad \{\text{ truth value is either } 0 \text{ or } 1\ \} \\
 & [\![\exists\, v.\ \varphi]\!]_{\mathcal{M},\,\sigma} = 1
\end{array}
$$

Soundness and semantic completeness

Theorem. $\vdash_{\mathrm{NJ}} \varphi$ (or $\vdash_{\mathrm{NK}} \varphi$) implies $\models \varphi$ for any first-order formula φ.

Completeness is trickier:

- classically it is a well-known result first proved by Gödel, but
- intuitionistically it has been shown to be unprovable by known methods unless we switch to a more sophisticated semantics.

Assuming completeness, we get the following undecidability result as a corollary of the negative answer to Hilbert's *Entscheidungsproblem* independently given by Church and Turing.

Theorem. Validity in classical first-order logic is undecidable.

Logical embedding

The fact that NK is obtained by extending NJ with the ($\neg\neg$E) rule suggests that intuitionistic logic is a sub-system of classical logic — some results developed in classical mathematics are constructive, while others are not.

The opposite view is possible, though: classical logic can be embedded into intuitionistic logic by the *Gödel–Gentzen negative translation*.

Assertion strength

In terms of provability, $\vdash_{\mathrm{NK}} \varphi \vee \neg\varphi$ does not assert that we can prove either φ or $\neg\varphi$, but that it cannot be the case that both φ and $\neg\varphi$ lead to contradiction — its strength is equivalent to that of $\vdash_{\mathrm{NJ}} \neg(\neg\varphi \wedge \neg\neg\varphi)$.

- A disjunctive proposition $\varphi \vee \psi$ in classical logic only amounts to $\neg(\neg\varphi^\circ \wedge \neg\psi^\circ)$ in intuitionistic logic.
- An existential statement $\exists\, v.\ \varphi$ in classical logic only amounts to $\neg\forall\, v.\ \neg\varphi^\circ$ in intuitionistic logic.
- An atomic proposition $p\ t_1 \ldots t_n$ in classical logic only amounts to the assertion that the opposite is impossible, i.e., $\neg\neg(p\ t_1 \ldots t_n)$.

As for $\bot$, '$\wedge$', '$\rightarrow$', and '$\forall$', their strength are the same in classical and intuitionistic logic.

Gödel–Gentzen negative translation

Definition. Given a signature $\mathcal{S} = (\mathcal{P}, \mathcal{F})$, the *Gödel–Gentzen negative translation* $\_^\circ : \textsc{Form}_\mathcal{S} \to \textsc{Form}_\mathcal{S}$ is defined by

$$
\begin{array}{lcll}
\bot^\circ & = & \bot & \\
(p\ t_1 \ldots t_n)^\circ & = & \neg\neg(p\ t_1 \ldots t_n) & \text{for } p : \mathcal{P} \\
(\varphi \wedge \psi)^\circ & = & \varphi^\circ \wedge \psi^\circ & \\
(\varphi \vee \psi)^\circ & = & \neg(\neg\varphi^\circ \wedge \neg\psi^\circ) & \\
(\varphi \to \psi)^\circ & = & \varphi^\circ \to \psi^\circ & \\
(\forall\, v.\ \varphi)^\circ & = & \forall\, v.\ \varphi^\circ & \\
(\exists\, v.\ \varphi)^\circ & = & \neg\forall\, v.\ \neg\varphi^\circ. &
\end{array}
$$

Gödel–Gentzen negative translation

Theorem. For any first-order formula φ and list Γ of first-order formulas, $\Gamma \vdash_{\mathrm{NK}} \varphi$ if and only if $\Gamma^\circ \vdash_{\mathrm{NJ}} \varphi^\circ$.

PROOF

1 ASSUME $\mathcal{S}$ signature, $\varphi : \textsc{Form}_{\mathcal{S}}$, $\Gamma : \textsc{List Form}_{\mathcal{S}}$, $\Gamma^\circ \vdash_{\mathrm{NJ}} \varphi^\circ$

PROVE $\Gamma \vdash_{\mathrm{NK}} \varphi$

PROOF

1.1 $\vdash_{\mathrm{NK}} \varphi \leftrightarrow \varphi^\circ$.

1.2 $\Gamma^\circ \vdash_{\mathrm{NK}} \varphi^\circ$. PROOF All rules of NJ are rules of NK.

1.3 QED. PROOF By 1.1, the “Gödel–Gentzen formulas” in 1.2 can be replaced by their untranslated versions under NK.

Gödel–Gentzen negative translation

2 ASSUME $\mathcal{S}$ signature

PROVE $\Gamma^\circ \vdash_{\mathrm{NJ}} \varphi^\circ$ for every $\varphi : \textsc{Form}_{\mathcal{S}}$ and $\Gamma : \textsc{List Form}_{\mathcal{S}}$ such that $\Gamma \vdash_{\mathrm{NK}} \varphi$.

PROOF

2.1 $\vdash_{\mathrm{NJ}} \psi^\circ \leftrightarrow \neg\neg\psi^\circ$ for any $\psi : \textsc{Form}_{\mathcal{S}}$. (For the Gödel–Gentzen formulas, double negation elimination is admissible in NJ.)

2.2 QED.

PROOF Induction on the derivation of $\Gamma \vdash_{\mathrm{NK}} \varphi$, using double negation elimination where necessary.

3 QED.

PROOF The two directions are proved in $\boxed{1}$ and $\boxed{2}$.

Gödel–Gentzen negative translation

Case ($\vee$E).

ASSUME $\Gamma : \textsc{List Form}_{\mathcal{S}}$, $\varphi, \psi, \vartheta : \textsc{Form}_{\mathcal{S}}$,
$d : \mathrm{NK}[\Gamma; \varphi \vee \psi]$, $d' : \mathrm{NJ}[\Gamma^\circ; \neg(\neg\varphi^\circ \wedge \neg\psi^\circ)]$,
$e : \mathrm{NK}[\Gamma, \varphi; \vartheta]$, $e' : \mathrm{NJ}[\Gamma^\circ, \varphi^\circ; \vartheta^\circ]$,
$f : \mathrm{NK}[\Gamma, \psi; \vartheta]$, $f' : \mathrm{NJ}[\Gamma^\circ, \psi^\circ; \vartheta^\circ]$

PROVE $\Gamma^\circ \vdash_{\mathrm{NJ}} \vartheta^\circ$

PROOF (d'', e'', and f'' are suitably weakened versions of d', e', and f'.)

$$
\dfrac{d'' \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma^\circ, \neg\vartheta^\circ, \varphi^\circ \vdash \neg\vartheta^\circ} \qquad e''}{\Gamma^\circ, \neg\vartheta^\circ, \varphi^\circ \vdash \bot}\,(\to\mathrm{E})}{\Gamma^\circ, \neg\vartheta^\circ \vdash \neg\varphi^\circ}\,(\to\mathrm{I}) \qquad \dfrac{\dfrac{\dfrac{}{\Gamma^\circ, \neg\vartheta^\circ, \psi^\circ \vdash \neg\vartheta^\circ} \qquad f''}{\Gamma^\circ, \neg\vartheta^\circ, \psi^\circ \vdash \bot}\,(\to\mathrm{E})}{\Gamma^\circ, \neg\vartheta^\circ \vdash \neg\psi^\circ}\,(\to\mathrm{I})}{\Gamma^\circ, \neg\vartheta^\circ \vdash \neg\varphi^\circ \wedge \neg\psi^\circ}\,(\wedge\mathrm{I})}{\dfrac{\Gamma^\circ, \neg\vartheta^\circ \vdash \bot}{\dfrac{\Gamma^\circ \vdash \neg\neg\vartheta^\circ}{\Gamma^\circ \vdash \vartheta^\circ}\,\boxed{2.1}}\,(\to\mathrm{I})}\,(\to\mathrm{E})
$$

Classical logic as a sub-language of intuitionistic logic

We might view

- the language of classical logic as a convenient way of writing the Gödel–Gentzen formulas in intuitionistic logic, and
- NK as an abstraction with which we can write certain indirect proofs in NJ more easily as direct proofs.

Under this view, intuitionistic logic is in fact a richer language, which has stronger disjunction and existential quantification and thus can distinguish constructive theorems from non-constructive ones (whereas classical logic cannot).