1.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C} \vdash (\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C}}}{(\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C} \vdash \mathtt{A} \wedge \mathtt{B}}\,(\wedge\mathrm{EL})}{(\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C} \vdash \mathtt{A}}\,(\wedge\mathrm{EL}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{(\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C} \vdash (\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C}}}{(\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C} \vdash \mathtt{A} \wedge \mathtt{B}}\,(\wedge\mathrm{EL})}{(\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C} \vdash \mathtt{B}}\,(\wedge\mathrm{ER}) \qquad \dfrac{\dfrac{}{(\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C} \vdash (\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C}}}{(\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C} \vdash \mathtt{C}}\,(\wedge\mathrm{ER})}{(\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C} \vdash \mathtt{B} \wedge \mathtt{C}}\,(\wedge\mathrm{I})}{\dfrac{(\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C} \vdash \mathtt{A} \wedge (\mathtt{B} \wedge \mathtt{C})}{\vdash (\mathtt{A} \wedge \mathtt{B}) \wedge \mathtt{C} \rightarrow \mathtt{A} \wedge (\mathtt{B} \wedge \mathtt{C})}\,(\rightarrow\mathrm{I})}\,(\wedge\mathrm{I})$$

2.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\mathtt{A \wedge B \to C,\ A,\ B \vdash A \wedge B \to C}}
        \quad
        \dfrac{
          \dfrac{}{\mathtt{A \wedge B \to C,\ A,\ B \vdash A}}
          \quad
          \dfrac{}{\mathtt{A \wedge B \to C,\ A,\ B \vdash B}}
        }{\mathtt{A \wedge B \to C,\ A,\ B \vdash A \wedge B}}\,(\wedge\mathsf{I})
      }{\mathtt{A \wedge B \to C,\ A,\ B \vdash C}}\,(\to\mathsf{E})
    }{\mathtt{A \wedge B \to C,\ A \vdash B \to C}}\,(\to\mathsf{I})
  }{
    \dfrac{\mathtt{A \wedge B \to C \vdash A \to B \to C}}{\mathtt{\vdash (A \wedge B \to C) \to (A \to B \to C)}}\,(\to\mathsf{I})
  }\,(\to\mathsf{I})
  \quad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\mathtt{A \to B \to C,\ A \wedge B \vdash A \to B \to C}}
          \quad
          \dfrac{
            \dfrac{}{\mathtt{A \to B \to C,\ A \wedge B \vdash A \wedge B}}
          }{\mathtt{A \to B \to C,\ A \wedge B \vdash A}}\,(\wedge\mathsf{EL})
        }{\mathtt{A \to B \to C,\ A \wedge B \vdash B \to C}}\,(\to\mathsf{E})
        \quad
        \dfrac{
          \dfrac{}{\mathtt{A \to B \to C,\ A \wedge B \vdash A \wedge B}}
        }{\mathtt{A \to B \to C,\ A \wedge B \vdash B}}\,(\wedge\mathsf{ER})
      }{\mathtt{A \to B \to C,\ A \wedge B \vdash C}}\,(\to\mathsf{E})
    }{\mathtt{A \to B \to C \vdash A \wedge B \to C}}\,(\to\mathsf{I})
  }{\mathtt{\vdash (A \to B \to C) \to (A \wedge B \to C)}}\,(\to\mathsf{I})
}{\mathtt{\vdash (A \wedge B \to C) \leftrightarrow (A \to B \to C)}}\,(\wedge\mathsf{I})
$$

3.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$A \lor (B \land C) \vdash A \lor (B \land C)$}
\AxiomC{}
\UnaryInfC{$A \lor (B \land C), A \vdash A$}
\RightLabel{($\lor$IL)}
\UnaryInfC{$A \lor (B \land C), A \vdash A \lor B$}
\AxiomC{}
\UnaryInfC{$A \lor (B \land C), A \vdash A$}
\RightLabel{($\lor$IL)}
\UnaryInfC{$A \lor (B \land C), A \vdash A \lor C$}
\RightLabel{($\land$I)}
\BinaryInfC{$A \lor (B \land C), A \vdash (A \lor B) \land (A \lor C)$}
\AxiomC{}
\UnaryInfC{$A \lor (B \land C), B \land C \vdash B \land C$}
\RightLabel{($\land$EL)}
\UnaryInfC{$A \lor (B \land C), B \land C \vdash B$}
\RightLabel{($\lor$IR)}
\UnaryInfC{$A \lor (B \land C), B \land C \vdash A \lor B$}
\AxiomC{}
\UnaryInfC{$A \lor (B \land C), B \land C \vdash B \land C$}
\RightLabel{($\land$ER)}
\UnaryInfC{$A \lor (B \land C), B \land C \vdash C$}
\RightLabel{($\lor$IR)}
\UnaryInfC{$A \lor (B \land C), B \land C \vdash A \lor C$}
\RightLabel{($\land$I)}
\BinaryInfC{$A \lor (B \land C), B \land C \vdash (A \lor B) \land (A \lor C)$}
\RightLabel{($\lor$E)}
\TrinaryInfC{$A \lor (B \land C) \vdash (A \lor B) \land (A \lor C)$}
\RightLabel{($\to$I)}
\UnaryInfC{$\vdash A \lor (B \land C) \to (A \lor B) \land (A \lor C)$}
\AxiomC{}
\UnaryInfC{$(A \lor B) \land (A \lor C) \vdash (A \lor B) \land (A \lor C)$}
\RightLabel{($\land$EL)}
\UnaryInfC{$(A \lor B) \land (A \lor C) \vdash A \lor B$}
\AxiomC{}
\UnaryInfC{$(A \lor B) \land (A \lor C), A \vdash A$}
\RightLabel{($\lor$IL)}
\UnaryInfC{$(A \lor B) \land (A \lor C), A \vdash A \lor (B \land C)$}
\AxiomC{}
\UnaryInfC{$(A \lor B) \land (A \lor C), B \vdash (A \lor B) \land (A \lor C)$}
\RightLabel{($\land$ER)}
\UnaryInfC{$(A \lor B) \land (A \lor C), B \vdash A \lor C$}
\AxiomC{}
\UnaryInfC{$(A \lor B) \land (A \lor C), B, A \vdash A$}
\RightLabel{($\lor$IL)}
\UnaryInfC{$(A \lor B) \land (A \lor C), B, A \vdash A \lor (B \land C)$}
\AxiomC{}
\UnaryInfC{$(A \lor B) \land (A \lor C), B, C \vdash B$}
\AxiomC{}
\UnaryInfC{$(A \lor B) \land (A \lor C), B, C \vdash C$}
\RightLabel{($\land$I)}
\BinaryInfC{$(A \lor B) \land (A \lor C), B, C \vdash B \land C$}
\RightLabel{($\lor$IR)}
\UnaryInfC{$(A \lor B) \land (A \lor C), B, C \vdash A \lor (B \land C)$}
\RightLabel{($\lor$E)}
\TrinaryInfC{$(A \lor B) \land (A \lor C), B \vdash A \lor (B \land C)$}
\RightLabel{($\lor$E)}
\TrinaryInfC{$(A \lor B) \land (A \lor C) \vdash A \lor (B \land C)$}
\RightLabel{($\to$I)}
\UnaryInfC{$\vdash (A \lor B) \land (A \lor C) \to A \lor (B \land C)$}
\RightLabel{($\land$I)}
\BinaryInfC{$\vdash A \lor (B \land C) \leftrightarrow (A \lor B) \land (A \lor C)$}
\end{prooftree}
$$

4.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A \vee B),\, A \vdash \neg(A \vee B)} \quad \dfrac{\dfrac{}{\neg(A \vee B),\, A \vdash A}}{\neg(A \vee B),\, A \vdash A \vee B}\,(\vee IL)}{\neg(A \vee B),\, A \vdash \bot}\,(\to E)}{\neg(A \vee B) \vdash \neg A}\,(\to I)$$

$$\dfrac{\dfrac{\dfrac{}{\neg(A \vee B),\, B \vdash \neg(A \vee B)} \quad \dfrac{\dfrac{}{\neg(A \vee B),\, B \vdash B}}{\neg(A \vee B),\, B \vdash A \vee B}\,(\vee IR)}{\neg(A \vee B),\, B \vdash \bot}\,(\to E)}{\neg(A \vee B) \vdash \neg B}\,(\to I)$$

$$\dfrac{\dfrac{\neg(A \vee B) \vdash \neg A \quad \neg(A \vee B) \vdash \neg B}{\neg(A \vee B) \vdash \neg A \wedge \neg B}\,(\wedge I)}{\vdash \neg(A \vee B) \to \neg A \wedge \neg B}\,(\to I)$$

$$\dfrac{\dfrac{\dfrac{}{\neg A \wedge \neg B,\, A \vee B,\, A \vdash \neg A \wedge \neg B}}{\neg A \wedge \neg B,\, A \vee B,\, A \vdash \neg A}\,(\wedge EL) \quad \dfrac{}{\neg A \wedge \neg B,\, A \vee B,\, A \vdash A}}{\neg A \wedge \neg B,\, A \vee B,\, A \vdash \bot}\,(\to E)$$

$$\dfrac{\dfrac{\dfrac{}{\neg A \wedge \neg B,\, A \vee B,\, B \vdash \neg A \wedge \neg B}}{\neg A \wedge \neg B,\, A \vee B,\, B \vdash \neg B}\,(\wedge ER) \quad \dfrac{}{\neg A \wedge \neg B,\, A \vee B,\, B \vdash B}}{\neg A \wedge \neg B,\, A \vee B,\, B \vdash \bot}\,(\to E)$$

$$\dfrac{\dfrac{\dfrac{\neg A \wedge \neg B,\, A \vee B \vdash A \vee B \quad \neg A \wedge \neg B,\, A \vee B,\, A \vdash \bot \quad \neg A \wedge \neg B,\, A \vee B,\, B \vdash \bot}{\neg A \wedge \neg B,\, A \vee B \vdash \bot}\,(\vee E)}{\neg A \wedge \neg B \vdash \neg(A \vee B)}\,(\to I)}{\vdash \neg A \wedge \neg B \to \neg(A \vee B)}\,(\to I)$$

$$\dfrac{\vdash \neg(A \vee B) \to \neg A \wedge \neg B \quad \vdash \neg A \wedge \neg B \to \neg(A \vee B)}{\vdash \neg(A \vee B) \leftrightarrow \neg A \wedge \neg B}\,(\wedge I)$$

5.

$$\dfrac{\dfrac{\dfrac{}{\neg(\mathtt{A}\lor\neg\mathtt{A})\vdash\neg(\mathtt{A}\lor\neg\mathtt{A})} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\mathtt{A}\lor\neg\mathtt{A}),\ \mathtt{A}\vdash\neg(\mathtt{A}\lor\neg\mathtt{A})} \qquad \dfrac{\dfrac{}{\neg(\mathtt{A}\lor\neg\mathtt{A}),\ \mathtt{A}\vdash\mathtt{A}}}{\neg(\mathtt{A}\lor\neg\mathtt{A}),\ \mathtt{A}\vdash\mathtt{A}\lor\neg\mathtt{A}}\,(\lor\text{IL})}{\neg(\mathtt{A}\lor\neg\mathtt{A}),\ \mathtt{A}\vdash\bot}\,(\to\text{E})}{\neg(\mathtt{A}\lor\neg\mathtt{A})\vdash\neg\mathtt{A}}\,(\to\text{I})}{\neg(\mathtt{A}\lor\neg\mathtt{A})\vdash\mathtt{A}\lor\neg\mathtt{A}}\,(\lor\text{IR})}{\neg(\mathtt{A}\lor\neg\mathtt{A})\vdash\bot}\,(\to\text{E})}{\vdash\neg\neg(\mathtt{A}\lor\neg\mathtt{A})}\,(\to\text{I})$$

6.

ASSUME $\Gamma : \textsc{List Prop},\ \ \varphi, \psi, \vartheta : \textsc{Prop},\ \ d : \mathrm{NJ}[\Gamma; \varphi \wedge \psi],\ \ e : \mathrm{NJ}[\Gamma, \varphi, \psi; \vartheta]$

PROVE $\Gamma \vdash_{\mathrm{NJ}} \vartheta$

PROOF

$$\dfrac{\dfrac{\dfrac{\dfrac{e}{\Gamma,\ \varphi \vdash \psi \to \vartheta}\ (\to\mathrm{I})}{\Gamma \vdash \varphi \to \psi \to \vartheta}\ (\to\mathrm{I}) \qquad \dfrac{d}{\Gamma \vdash \varphi}\ (\wedge\mathrm{EL})}{\Gamma \vdash \psi \to \vartheta}\ (\to\mathrm{E}) \qquad \dfrac{d}{\Gamma \vdash \psi}\ (\wedge\mathrm{ER})}{\Gamma \vdash \vartheta}\ (\to\mathrm{E})$$

7.

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\forall x.\ P\ x \land Q\ x \vdash \forall x.\ P\ x \land Q\ x}
}{\forall x.\ P\ x \land Q\ x \vdash P\ x \land Q\ x}\ (\forall E)
}{\forall x.\ P\ x \land Q\ x \vdash P\ x}\ (\land EL)
}{\forall x.\ P\ x \land Q\ x \vdash \forall x.\ P\ x}\ (\forall I)
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\forall x.\ P\ x \land Q\ x \vdash \forall x.\ P\ x \land Q\ x}
}{\forall x.\ P\ x \land Q\ x \vdash P\ x \land Q\ x}\ (\forall E)
}{\forall x.\ P\ x \land Q\ x \vdash Q\ x}\ (\land ER)
}{\forall x.\ P\ x \land Q\ x \vdash \forall x.\ Q\ x}\ (\forall I)
}{\forall x.\ P\ x \land Q\ x \vdash (\forall x.\ P\ x) \land (\forall x.\ Q\ x)}\ (\land I)
}{\vdash (\forall x.\ P\ x \land Q\ x) \to (\forall x.\ P\ x) \land (\forall x.\ Q\ x)}\ (\to I)
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{(\forall x.\ P\ x) \land (\forall x.\ Q\ x) \vdash (\forall x.\ P\ x) \land (\forall x.\ Q\ x)}
}{(\forall x.\ P\ x) \land (\forall x.\ Q\ x) \vdash \forall x.\ P\ x}\ (\land EL)
}{(\forall x.\ P\ x) \land (\forall x.\ Q\ x) \vdash P\ x}\ (\forall E)
\qquad
\dfrac{
\dfrac{
\dfrac{}{(\forall x.\ P\ x) \land (\forall x.\ Q\ x) \vdash (\forall x.\ P\ x) \land (\forall x.\ Q\ x)}
}{(\forall x.\ P\ x) \land (\forall x.\ Q\ x) \vdash \forall x.\ Q\ x}\ (\land ER)
}{(\forall x.\ P\ x) \land (\forall x.\ Q\ x) \vdash Q\ x}\ (\forall E)
}{(\forall x.\ P\ x) \land (\forall x.\ Q\ x) \vdash P\ x \land Q\ x}\ (\land I)
}{(\forall x.\ P\ x) \land (\forall x.\ Q\ x) \vdash \forall x.\ P\ x \land Q\ x}\ (\forall I)
}{\vdash (\forall x.\ P\ x) \land (\forall x.\ Q\ x) \to (\forall x.\ P\ x \land Q\ x)}\ (\to I)
}{\vdash (\forall x.\ P\ x \land Q\ x) \leftrightarrow (\forall x.\ P\ x) \land (\forall x.\ Q\ x)}\ (\land I)
$$

8.

$$\dfrac{\dfrac{}{(\exists x.\ P\ x) \to Q,\ P\ x \vdash (\exists x.\ P\ x) \to Q} \qquad \dfrac{\dfrac{}{(\exists x.\ P\ x) \to Q,\ P\ x \vdash P\ x}}{(\exists x.\ P\ x) \to Q,\ P\ x \vdash \exists x.\ P\ x}\ (\exists I)}{(\exists x.\ P\ x) \to Q,\ P\ x \vdash Q}\ (\to E)$$

$$\dfrac{\dfrac{\dfrac{(\exists x.\ P\ x) \to Q,\ P\ x \vdash Q}{(\exists x.\ P\ x) \to Q \vdash P\ x \to Q}\ (\to I)}{(\exists x.\ P\ x) \to Q \vdash \forall x.\ P\ x \to Q}\ (\forall I)}{\vdash ((\exists x.\ P\ x) \to Q) \to \forall x.\ P\ x \to Q}\ (\to I)$$

$$\dfrac{\dfrac{\dfrac{}{\forall x.\ P\ x \to Q,\ \exists x.\ P\ x,\ P\ x \vdash \forall x.\ P\ x \to Q}}{\forall x.\ P\ x \to Q,\ \exists x.\ P\ x,\ P\ x \vdash P\ x \to Q}\ (\forall E) \qquad \dfrac{}{\forall x.\ P\ x \to Q,\ \exists x.\ P\ x,\ P\ x \vdash P\ x}}{\forall x.\ P\ x \to Q,\ \exists x.\ P\ x,\ P\ x \vdash Q}\ (\to E)$$

$$\dfrac{\dfrac{}{\forall x.\ P\ x \to Q,\ \exists x.\ P\ x \vdash \exists x.\ P\ x} \qquad \forall x.\ P\ x \to Q,\ \exists x.\ P\ x,\ P\ x \vdash Q}{\forall x.\ P\ x \to Q,\ \exists x.\ P\ x \vdash Q}\ (\exists E)$$

$$\dfrac{\dfrac{\forall x.\ P\ x \to Q,\ \exists x.\ P\ x \vdash Q}{\forall x.\ P\ x \to Q \vdash (\exists x.\ P\ x) \to Q}\ (\to I)}{\vdash (\forall x.\ P\ x \to Q) \to (\exists x.\ P\ x) \to Q}\ (\to I)$$

$$\dfrac{\vdash ((\exists x.\ P\ x) \to Q) \to \forall x.\ P\ x \to Q \qquad \vdash (\forall x.\ P\ x \to Q) \to (\exists x.\ P\ x) \to Q}{\vdash ((\exists x.\ P\ x) \to Q) \leftrightarrow (\forall x.\ P\ x \to Q)}\ (\wedge I)$$

9.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x.\, P\ x,\ \forall x.\, \neg(P\ x) \vdash \exists x.\, P\ x} \qquad \dfrac{\dfrac{\dfrac{}{\exists x.\, P\ x,\ \forall x.\, \neg(P\ x),\ P\ x \vdash \forall x.\, \neg(P\ x)}}{\exists x.\, P\ x,\ \forall x.\, \neg(P\ x),\ P\ x \vdash \neg(P\ x)}\,(\forall E) \qquad \dfrac{}{\exists x.\, P\ x,\ \forall x.\, \neg(P\ x),\ P\ x \vdash P\ x}}{\exists x.\, P\ x,\ \forall x.\, \neg(P\ x),\ P\ x \vdash \bot}\,(\to E)}{\exists x.\, P\ x,\ \forall x.\, \neg(P\ x) \vdash \bot}\,(\exists E)}{\exists x.\, P\ x \vdash \neg(\forall x.\, \neg(P\ x))}\,(\to I)}{\vdash (\exists x.\, P\ x) \to \neg(\forall x.\, \neg(P\ x))}\,(\to I)$$

10.

$$\dfrac{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \textit{transitivity}}{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \forall \mathtt{y}.\forall \mathtt{z}.\ \mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{y}\wedge\mathtt{y}\equiv\mathtt{z}\rightarrow\mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{z}}\ (\forall\mathsf{E})$$

$$\dfrac{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \forall \mathtt{y}.\forall \mathtt{z}.\ \mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{y}\wedge\mathtt{y}\equiv\mathtt{z}\rightarrow\mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{z}}{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \forall \mathtt{z}.\ \mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ (x+zero)}\wedge\mathtt{suc\ (x+zero)}\equiv\mathtt{z}\rightarrow\mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{z}}\ (\forall\mathsf{E})$$

$$\dfrac{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \forall \mathtt{z}.\ \mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ (x+zero)}\wedge\mathtt{suc\ (x+zero)}\equiv\mathtt{z}\rightarrow\mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{z}}{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ (x+zero)}\wedge\mathtt{suc\ (x+zero)}\equiv\mathtt{suc\ x}\rightarrow\mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ x}}\ (\forall\mathsf{E})$$

$$\dfrac{\dfrac{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \textit{additionS}}{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \forall\mathtt{y}.\ \mathtt{suc\ x}+\mathtt{y}\equiv\mathtt{suc\ (x+y)}}\ (\forall\mathsf{E})}{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ (x+zero)}}\ (\forall\mathsf{E})$$

$$\dfrac{\dfrac{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \textit{congruence}}{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \forall\mathtt{y}.\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{y}\rightarrow\mathtt{suc\ (x+zero)}\equiv\mathtt{suc\ y}}\ (\forall\mathsf{E})}{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\rightarrow\mathtt{suc\ (x+zero)}\equiv\mathtt{suc\ x}}\ (\forall\mathsf{E})$$

$$\dfrac{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\rightarrow\mathtt{suc\ (x+zero)}\equiv\mathtt{suc\ x}\qquad \mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}}{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{suc\ (x+zero)}\equiv\mathtt{suc\ x}}\ (\rightarrow\mathsf{E})$$

$$\dfrac{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ (x+zero)}\qquad \mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{suc\ (x+zero)}\equiv\mathtt{suc\ x}}{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ (x+zero)}\wedge\mathtt{suc\ (x+zero)}\equiv\mathtt{suc\ x}}\ (\wedge\mathsf{I})$$

$$\dfrac{\begin{array}{c}\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ (x+zero)}\wedge\mathtt{suc\ (x+zero)}\equiv\mathtt{suc\ x}\rightarrow\mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ x}\\ \mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ (x+zero)}\wedge\mathtt{suc\ (x+zero)}\equiv\mathtt{suc\ x}\end{array}}{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ x}}\ (\rightarrow\mathsf{E})$$

$$\dfrac{\dfrac{\mathbf{HA},\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\vdash \mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ x}}{\mathbf{HA}\vdash \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\rightarrow\mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ x}}\ (\rightarrow\mathsf{I})}{\mathbf{HA}\vdash \forall\mathtt{x}.\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\rightarrow\mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ x}}\ (\forall\mathsf{I})$$

$$\dfrac{\mathbf{HA}\vdash \textit{additionZ}}{\mathbf{HA}\vdash \mathtt{zero}+\mathtt{zero}\equiv\mathtt{zero}}\ (\forall\mathsf{E})$$

$$\dfrac{\mathbf{HA}\vdash \mathtt{zero}+\mathtt{zero}\equiv\mathtt{zero}\qquad \mathbf{HA}\vdash \forall\mathtt{x}.\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\rightarrow\mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ x}}{\mathbf{HA}\vdash (\mathtt{zero}+\mathtt{zero}\equiv\mathtt{zero})\wedge(\forall\mathtt{x}.\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\rightarrow\mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ x})}\ (\wedge\mathsf{I})$$

$$\dfrac{\mathbf{HA}\vdash \textit{induction}_{\mathtt{x+zero\equiv x},\ \mathtt{x}}\qquad \mathbf{HA}\vdash (\mathtt{zero}+\mathtt{zero}\equiv\mathtt{zero})\wedge(\forall\mathtt{x}.\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}\rightarrow\mathtt{suc\ x}+\mathtt{zero}\equiv\mathtt{suc\ x})}{\mathbf{HA}\vdash \forall\mathtt{x}.\ \mathtt{x}+\mathtt{zero}\equiv\mathtt{x}}\ (\rightarrow\mathsf{E})$$