Functional Programming
Exercise 3: Inductively Defined Functions

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1. Knowing that how addition on natural numbers can be defined, how does one define multiplication? Define a function $mul\ :\ int \rightarrow int \rightarrow int$ that performes multiplication, assuming both arguments are natural numbers. You may reuse $(+)$.

2. Define your version of the function $length : {}'a\ list$ that returns the length of a list (note that $[\,]$ has length 0).

3. Prove that *length* distributes into $(@)$:

$$length\ (xs\ @\ ys) = length\ xs + length\ ys$$

4. Prove: $sum \ll concat = sum \ll map\ sum$.

5. Prove: $take\ n\ xs\ @\ drop\ n\ xs = xs$, for all n and xs.

6. Define functions *inits* and *tails*, both of type $'a\ list \rightarrow {}'a\ list\ list$, such that the former returns all prefixes of a list, while the latter returns all suffixes of a list. E.g.

   - $inits\ [1; 2; 3] = [[\,]; [1]; [1; 2]; [1; 2; 3]]$
   - $tails\ [1; 2; 3] = [[1; 2; 3]; [2; 3]; [3]; [\,]]$

   Hint: Notice that $[\,]$ is a prefix (suffix) of any list. Thus both *inits* and *tails* always return a list containing $[\,]$. In particular, $inits\ [\,] = tails\ [\,] = [[\,]]$.

7. Define a function $fan :: {}'a \rightarrow {}'a\ list \rightarrow {}'a\ list\ list$ such that $fan\ x\ xs$ inserts x into the 0th, 1st…nth positions of xs, where n is the length of xs. For example:

$$fan\ 5\ [1; 2; 3; 4] = [[5; 1; 2; 3; 4]; [1; 5; 2; 3; 4]; [1; 2; 5; 3; 4]; [1; 2; 3; 5; 4]; [1; 2; 3; 4; 5]]$$

8. Define $perms :: {}'a\ list \rightarrow {}'a\ list\ list$ that returns all permutations of the input list. For example:

$$perms\ [1; 2; 3] = [[1; 2; 3]; [2; 1; 3]; [2; 3; 1]; [1; 3; 2]; [3; 1; 2]; [3; 2; 1]]$$

   You will need several auxiliary functions defined in the lectures and in the exercises.