Program Construction and Reasoning
Exercises (Part 2)

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Quantifications

1. An integer array $X[0..N)$ is given, where $N \geq 1$. Express the following sentences in a formal way:
    1. r is the sum of the elements of X.
    2. X is increasing.
    3. all values of X are distinct.
    4. r is the length of a longest constant segment of X.
    5. r is the maximum of the sums of the segments of X.
2. An integer array $X[0..N)$ is given, where $N \geq 1$. Express the following expressions in a natural language:
    1. $b \leftrightarrow (\forall i : 0 \leq i < N : X[i] \geq 0)$.
    2. $r = (\#k : 0 \leq k < N : (\forall i : 0 \leq i < k : X[i] < X[k]))$.
    3. $r = (\uparrow p, q : 0 \leq p \leq q \leq N \ \wedge \ (\forall i : p \leq i < q : X[i] > 0) : p - q)$.
    4. $r = (\#p, q : 0 \leq p < q < N : X[p] = 0 \ \wedge \ X[q] = 1)$.
    5. $s = (\uparrow p, q : 0 \leq p < q < N : X[p] + X[q])$.
    6. $b \leftrightarrow (\forall p, q : 0 \leq p \ \wedge \ 0 \leq q \ \wedge \ p + q = N - 1 : X[p] = X[q])$.

Taking Conjuncts as Invariants

3. Derive a program for the computation of square root.

$$
\begin{array}{l}
|[\ \mathbf{con}\ N : \mathit{int}\{N \geq 0\};\\
\quad \mathbf{var}\ x : \mathit{int};\\
\quad \mathit{squareroot}\\
\quad \{x^2 \leq N \ \wedge \ (x+1)^2 > N\}\\
]|
\end{array}
$$

Solution: Try using $x^2 \leq N$ as the invariant and $\neg((x+1)^2 > N)$ as the guard.

Replacing Constants by Variables

4. Derive a solution for:

$$\begin{array}{l} |[\ \textbf{con}\ N : int\{N \geq 0\}; a : \textbf{array}\ [0..N)\ \textbf{of}\ int; \\ \quad \textbf{var}\ r : int; \\ \quad S \\ \quad \{r = (\uparrow i : 0 \leq i < N : a[i])\} \\ ]|. \end{array}$$

5. Derive a solution for:

$$\begin{array}{l} |[\ \textbf{con}\ N, X : int\{N \geq 0\}; a : \textbf{array}\ [0..N)\ \textbf{of}\ int; \\ \quad \textbf{var}\ r : int; \\ \quad S \\ \quad \{r = (\Sigma i : 0 \leq i < N : a[i] \times X^i)\} \\ ]|. \end{array}$$

Solution: For efficiency, add a variable x and use the invariant:

$$r = (\Sigma i : 0 \leq i < n : a[i] \times X^i) \ \wedge \ x = X^n \ \wedge \ 0 \leq n \leq N.$$

Another possibility, however, is to define for $0 \leq n \leq N$:

$$k\ n \ = \ (\Sigma i : n \leq i < N : a[i] \times X^{i-n}),$$

use the invariant $r = k\ n \wedge 0 \leq n \leq N$, and decrement n in the loop.