Logic

Solutions to Homework for Lecture II

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These are possible solutions to the homework for the second lecture. Some questions can be answered in more than one way, so if your answer differs from mine that does not mean you are wrong. In fact, my solution might be wrong, in which case you should contact me as soon as possible.

1 Sequent Calculus for Propositional Logic

Give derivations of the following sequents:

1. $\vdash (P \wedge Q \to R) \to (P \to Q \to R)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P, Q \vdash P, R \qquad P, Q \vdash Q, R}{P, Q \vdash P \wedge Q, R}\,(\wedge\mathrm{R}) \qquad P, Q, R \vdash R}{P \wedge Q \to R, P, Q \vdash R}\,(\to\mathrm{L})}{P \wedge Q \to R, P \vdash Q \to R}\,(\to\mathrm{R})}{P \wedge Q \to R \vdash P \to Q \to R}\,(\to\mathrm{R})}{\vdash (P \wedge Q \to R) \to (P \to Q \to R)}\,(\to\mathrm{R})
$$

2. $\vdash P \to \neg\neg P$

$$
\dfrac{\dfrac{\dfrac{P \vdash P}{P, \neg P \vdash}\,(\neg\mathrm{L})}{P \vdash \neg\neg P}\,(\neg\mathrm{R})}{\vdash P \to \neg\neg P}\,(\to\mathrm{R})
$$

3. $\vdash (P \to Q) \vee (Q \to P)$

$$
\dfrac{\dfrac{\dfrac{P, Q \vdash P, Q}{P \vdash Q, Q \to P}\,(\to\mathrm{R})}{\vdash P \to Q, Q \to P}\,(\to\mathrm{R})}{\vdash (P \to Q) \vee (Q \to P)}\,(\vee\mathrm{R})
$$

4. $\vdash P \vee \neg P$

$$
\begin{prooftree}
\AxiomC{$P \vdash P$}
\LeftLabel{($\neg$R)}
\UnaryInfC{$\vdash P, \neg P$}
\LeftLabel{($\vee$R)}
\UnaryInfC{$\vdash P \vee \neg P$}
\end{prooftree}
$$

5. $\vdash \neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$

$$
\begin{prooftree}
\AxiomC{$P, Q \vdash P$}
\AxiomC{$P, Q \vdash Q$}
\LeftLabel{($\wedge$R)}
\BinaryInfC{$P, Q \vdash P \wedge Q$}
\LeftLabel{($\neg$L)}
\UnaryInfC{$\neg(P \wedge Q), P, Q \vdash$}
\LeftLabel{($\neg$R)}
\UnaryInfC{$\neg(P \wedge Q), P \vdash \neg Q$}
\LeftLabel{($\neg$R)}
\UnaryInfC{$\neg(P \wedge Q) \vdash \neg P, \neg Q$}
\LeftLabel{($\vee$R)}
\UnaryInfC{$\neg(P \wedge Q) \vdash \neg P \vee \neg Q$}
\LeftLabel{($\to$R)}
\UnaryInfC{$\vdash \neg(P \wedge Q) \to \neg P \vee \neg Q$}
\AxiomC{$P, Q \vdash P$}
\LeftLabel{($\wedge$L)}
\UnaryInfC{$P \wedge Q \vdash P$}
\LeftLabel{($\neg$L)}
\UnaryInfC{$\neg P, P \wedge Q \vdash$}
\AxiomC{$P, Q \vdash Q$}
\LeftLabel{($\wedge$L)}
\UnaryInfC{$P \wedge Q \vdash Q$}
\LeftLabel{($\neg$L)}
\UnaryInfC{$\neg Q, P \wedge Q \vdash$}
\LeftLabel{($\vee$L)}
\BinaryInfC{$\neg P \vee \neg Q, P \wedge Q \vdash$}
\LeftLabel{($\neg$R)}
\UnaryInfC{$\neg P \vee \neg Q \vdash \neg(P \wedge Q)$}
\LeftLabel{($\to$R)}
\UnaryInfC{$\vdash \neg P \vee \neg Q \to \neg(P \wedge Q)$}
\LeftLabel{($\wedge$R)}
\BinaryInfC{$\vdash \neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$}
\end{prooftree}
$$

2 Sequent Calculus for First-order Logic

Hint: To solve the problems below, you may make use of the fact that $\varphi[x/x] \equiv \varphi$ for any formula φ and any variable x.

1. Give a derivation of $\vdash \varphi \leftrightarrow (\forall x.\varphi)$, where φ is a formula such that $x \notin \mathrm{FV}(\varphi)$. Which part of the derivation fails when this condition is not satisfied?

$$
\begin{prooftree}
\AxiomC{$\varphi \vdash \varphi$}
\LeftLabel{($\forall$R)}
\UnaryInfC{$\varphi \vdash \forall x.\varphi$}
\LeftLabel{($\to$R)}
\UnaryInfC{$\vdash \varphi \to (\forall x.\varphi)$}
\AxiomC{$\varphi \vdash \varphi$}
\LeftLabel{($\forall$L)}
\UnaryInfC{$\forall x.\varphi \vdash \varphi$}
\LeftLabel{($\to$R)}
\UnaryInfC{$\vdash (\forall x.\varphi) \to \varphi$}
\LeftLabel{($\wedge$R)}
\BinaryInfC{$\vdash \varphi \leftrightarrow (\forall x.\varphi)$}
\end{prooftree}
$$

The application of ($\forall$R) is only valid because $x \notin \mathrm{FV}(\varphi)$.

2. Can you give a derivation of $\vdash (\forall x.\varphi) \to (\exists x.\varphi)$ for any formula φ? Would you accept this inference step in a mathematical proof? Why or why not?

$$
\begin{prooftree}
\AxiomC{$\varphi \vdash \varphi$}
\LeftLabel{($\forall$L)}
\UnaryInfC{$\forall x.\varphi \vdash \varphi$}
\LeftLabel{($\exists$R)}
\UnaryInfC{$\forall x.\varphi \vdash \exists x.\varphi$}
\LeftLabel{($\to$R)}
\UnaryInfC{$\vdash (\forall x.\varphi) \to (\exists x.\varphi)$}
\end{prooftree}
$$

In a mathematical proof, this derivation might seem suspiciuos: on an empty domain, any formula $\forall x.\varphi$ is vacuously held to be true, but any $\exists x.\varphi$ would be false.

However, in logic we always assume that there is at least one element in the domain.

3. Show that $\vdash_{\text{LK}} \neg(\exists x.\neg\varphi) \to (\forall x.\varphi)$ for any formula φ.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi}{\vdash \varphi, \neg\varphi}\,(\neg\text{R})}{\vdash \varphi, \exists x.\neg\varphi}\,(\exists\text{R})}{\vdash \forall x.\varphi, \exists x.\neg\varphi}\,(\forall\text{R})}{\neg\exists x.\neg\varphi \vdash \forall x.\varphi}\,(\neg\text{L})}{\vdash \neg(\exists x.\neg\varphi) \to (\forall x.\varphi)}\,(\to\text{R})$$

Note that the side condition of $(\forall\text{R})$ is satisfied, since $x \notin \text{FV}(\exists x.\neg\varphi)$.

4. Show that $\vdash_{\text{LK}} (\exists x.\forall y.\varphi) \to (\forall y.\exists x.\varphi)$ for any formula φ.

Give a structure $\mathcal{M}$ and a formula φ with free variables x and y such that $\mathcal{M} \models \forall y.\exists x.\varphi$, but $\mathcal{M} \not\models \exists x.\forall y.\varphi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi}{\varphi \vdash \exists x.\varphi}\,(\exists\text{R})}{\forall y.\varphi \vdash \exists x.\varphi}\,(\forall\text{L})}{\forall y.\varphi \vdash \forall y.\exists x.\varphi}\,(\forall\text{R})}{\forall x.\forall y.\varphi \vdash \forall y.\exists x.\varphi}\,(\exists\text{L})$$

Reading top down, we first apply $(\exists\text{R})$ and $(\forall\text{L})$ because they do not have any side conditions. When we later apply $(\forall\text{R})$ and $(\exists\text{L})$, their side conditions are satisfied, since x and y are already bound.

For the second half of the problem, take the structure $\mathcal{M}$ for Σ_{ar} introduced in Example 10 on page 18 of the lecture notes, and choose φ to be $x \doteq s(y)$.

We first show $\mathcal{M} \vdash \forall y.\exists x.x \doteq s(y)$. Given a natural number n and an arbitrary variable assignment σ, we have

$$\mathcal{M}, \sigma[y := n][x := n+1] \models x \doteq s(y)$$

and hence $\mathcal{M}, \sigma[y := n] \models \exists x.x \doteq s(y)$. Since n was arbitrary we get $\mathcal{M}, \sigma \models \forall y.\exists x.x \doteq s(y)$, and since σ was arbitrary also

$$\mathcal{M} \models \forall y.\exists x.x \doteq s(y).$$

However, we have $\mathcal{M} \not\models \exists x.\forall y.x \doteq s(y)$. Indeed, assume we had $\mathcal{M} \models \exists x.\forall y.x \doteq s(y)$. Then for any variable assignment σ there would have to be a natural number n such that $\mathcal{M}, \sigma[x := n] \models \forall y.x \doteq s(y)$; but obviously $\mathcal{M}, \sigma[x := n][y := n] \not\models x \doteq s(y)$, so this cannot be true.