Logic

Solutions to Homework for Lecture I

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These are possible solutions to the homework for the first lecture. Some questions can be answered in more than one way, so if your answer differs from mine that does not mean you are wrong. In fact, my solution might be wrong, in which case you should contact me as soon as possible.

1. Show the following equivalence (often called *Consensus Theorem*) by drawing up a truth table:

$$P \wedge Q \;\vee\; \neg P \wedge R \;\vee\; Q \wedge R \Leftrightarrow P \wedge Q \;\vee\; \neg P \wedge R$$

Let $\varphi_1 := P \wedge Q \;\vee\; \neg P \wedge R \;\vee\; Q \wedge R$ *and* $\varphi_2 := P \wedge Q \;\vee\; \neg P \wedge R$*. Then we have:*

P	Q	R	$P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	φ_1	φ_2
F	F	F	F	F	F	F	F
F	F	T	F	T	F	T	T
F	T	F	F	F	F	F	F
F	T	T	F	T	T	T	T
T	F	F	F	F	F	F	F
T	F	T	F	F	F	F	F
T	T	F	T	F	F	T	T
T	T	T	T	F	T	T	T

2. The connective $\overline{\vee}$ ("nor") is defined by $\varphi \,\overline{\vee}\, \psi := \neg(\varphi \vee \psi)$.

   (a) Draw a truth table for $P \,\overline{\vee}\, Q$.

P	Q	$P \,\overline{\vee}\, Q$
F	F	T
F	T	F
T	F	F
T	T	F

   (b) Find formulas $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ with $\overline{\vee}$ as their only connective such that

      i. $\varphi_1 \Leftrightarrow \neg P$

      ii. $\varphi_2 \Leftrightarrow P \vee Q$

iii. $\varphi_3 \Leftrightarrow P \wedge Q$

iv. $\varphi_4 \Leftrightarrow P \rightarrow Q$

You do not have to provide truth tables.

We can choose $\varphi_1 := P \,\overline{\vee}\, P$, $\varphi_2 := (P \,\overline{\vee}\, Q) \,\overline{\vee}\, (P \,\overline{\vee}\, Q)$, $\varphi_3 := (P \,\overline{\vee}\, P) \,\overline{\vee}\, (Q \,\overline{\vee}\, Q)$ *and* $\varphi_4 := ((P \,\overline{\vee}\, P) \,\overline{\vee}\, Q) \,\overline{\vee}\, ((P \,\overline{\vee}\, P) \,\overline{\vee}\, Q)$.

Is $\{\overline{\vee}\}$ a functionally complete set?

Not according to our definition, since we cannot build a formula without propositional letters from it.

3. Prove the following tautologies using calculational reasoning with the laws on pg. 12 of the lecture notes (you may additionally want to use the law $P \vee \bot \leftrightarrow P$ proved in the lecture):

(a) $P \wedge P \leftrightarrow P$

$$
\begin{array}{ll}
 & \underline{P \wedge P} \\
\Leftrightarrow & \qquad \{ \textit{ Golden Rule } \} \\
 & \underline{P \leftrightarrow P} \leftrightarrow P \vee P \\
\Leftrightarrow & \qquad \{ \textit{ Unfolding } \top \} \\
 & \underline{\top \leftrightarrow P \vee P} \\
\Leftrightarrow & \qquad \{ \textit{ Unfolding } \top \} \\
 & \underline{P \vee P} \\
\Leftrightarrow & \qquad \{ \textit{ Idempotency of } \vee \} \\
 & P
\end{array}
$$

(b) $P \wedge Q \leftrightarrow Q \wedge P$

$$
\begin{array}{ll}
 & \underline{P \wedge Q} \\
\Leftrightarrow & \qquad \{ \textit{ Golden Rule } \} \\
 & \underline{P \leftrightarrow Q} \leftrightarrow P \vee Q \\
\Leftrightarrow & \qquad \{ \textit{ Symmetry of } \leftrightarrow \} \\
 & Q \leftrightarrow P \leftrightarrow \underline{P \vee Q} \\
\Leftrightarrow & \qquad \{ \textit{ Symmetry of } \vee \} \\
 & \underline{Q \leftrightarrow P \leftrightarrow Q \vee P} \\
\Leftrightarrow & \qquad \{ \textit{ Golden Rule } \} \\
 & Q \wedge P
\end{array}
$$

(c) (*extra credit*) $P \wedge (Q \wedge R) \leftrightarrow (P \wedge Q) \wedge R$

$$
\begin{array}{ll}
 & \underline{P \wedge (Q \wedge R)} \\
\Leftrightarrow & \qquad \{\ \textit{Golden Rule}\ \} \\
 & P \leftrightarrow \underline{Q \wedge R} \leftrightarrow P \vee \underline{Q \wedge R} \\
\Leftrightarrow & \qquad \{\ \textit{Golden Rule (twice)}\ \} \\
 & P \leftrightarrow Q \leftrightarrow R \leftrightarrow Q \vee R \leftrightarrow \underline{P \vee (Q \leftrightarrow R \leftrightarrow Q \vee R)} \\
\Leftrightarrow & \qquad \{\ \textit{Associativity of } \vee \textit{ (twice)}\ \} \\
 & P \leftrightarrow Q \leftrightarrow R \leftrightarrow Q \vee R \leftrightarrow P \vee Q \leftrightarrow P \vee R \leftrightarrow P \vee Q \vee R \\
\Leftrightarrow & \qquad \{\ \textit{Symmetry and associativity of } \leftrightarrow \textit{ (several times)}\ \} \\
 & P \leftrightarrow Q \leftrightarrow P \vee Q \leftrightarrow R \leftrightarrow \underline{P \vee R \leftrightarrow Q \vee R \leftrightarrow P \vee Q \vee R} \\
\Leftrightarrow & \qquad \{\ \textit{Associativity of } \vee \textit{ (twice)}\ \} \\
 & \underline{P \leftrightarrow Q \leftrightarrow P \vee Q} \leftrightarrow R \leftrightarrow \underline{(P \leftrightarrow Q \leftrightarrow P \vee Q)} \vee R \\
\Leftrightarrow & \qquad \{\ \textit{Golden Rule (twice)}\ \} \\
 & \underline{P \wedge Q \leftrightarrow R \leftrightarrow P \wedge Q \vee R} \\
\Leftrightarrow & \qquad \{\ \textit{Golden Rule}\ \} \\
 & (P \wedge Q) \wedge R
\end{array}
$$

(d) $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$ (Hint: start with the right hand side)

$$
\begin{array}{ll}
 & \underline{\neg P \wedge \neg Q} \\
\Leftrightarrow & \qquad \{ \textit{ Golden Rule } \} \\
 & \neg P \leftrightarrow \neg Q \leftrightarrow \neg P \vee \underline{\neg Q} \\
\Leftrightarrow & \qquad \{ \textit{ Unfolding } \neg \ \} \\
 & \neg P \leftrightarrow \neg Q \leftrightarrow \underline{\neg P \vee (Q \leftrightarrow \bot)} \\
\Leftrightarrow & \qquad \{ \textit{ Distributivity of } \vee \ \} \\
 & \neg P \leftrightarrow \neg Q \leftrightarrow \neg P \vee Q \leftrightarrow \underline{\neg P \vee \bot} \\
\Leftrightarrow & \qquad \{ \ P \vee \bot \leftrightarrow P \textit{ and symmetry of } \leftrightarrow \ \} \\
 & \underline{\neg P \leftrightarrow \neg P} \leftrightarrow \neg Q \leftrightarrow \neg P \vee Q \\
\Leftrightarrow & \qquad \{ \textit{ Unfolding } \top \textit{ twice } \} \\
 & \neg Q \leftrightarrow \underline{\neg P} \vee Q \\
\Leftrightarrow & \qquad \{ \textit{ Unfolding } \neg \ \} \\
 & \neg Q \leftrightarrow \underline{(P \leftrightarrow \bot) \vee Q} \\
\Leftrightarrow & \qquad \{ \textit{ Distributivity of } \vee \ \} \\
 & \neg Q \leftrightarrow P \vee Q \leftrightarrow \underline{\bot \vee Q} \\
\Leftrightarrow & \qquad \{ \ P \vee \bot \leftrightarrow P \ \} \\
 & \underline{\neg Q} \leftrightarrow P \vee Q \leftrightarrow Q \\
\Leftrightarrow & \qquad \{ \textit{ Unfolding } \neg \textit{ and symmetry of } \leftrightarrow \ \} \\
 & \bot \leftrightarrow \underline{Q \leftrightarrow Q} \leftrightarrow P \vee Q \\
\Leftrightarrow & \qquad \{ \textit{ Unfolding } \top \textit{ twice } \} \\
 & \underline{\bot \leftrightarrow P \vee Q} \\
\Leftrightarrow & \qquad \{ \textit{ Unfolding } \neg \ \} \\
 & \neg(P \vee Q)
\end{array}
$$

4. (*extra credit*) Back on the island of knights and knaves, inhabitant A says “It is not the case that I am a knight if B says so.” Using calculational logic, what can you deduce about A and B?

    A's statement can be translated into logic as

$$\neg((B \leftrightarrow A) \rightarrow A)$$

We simplify as follows:

$$
\begin{array}{ll}
 & \underline{A \leftrightarrow \neg((B \leftrightarrow A) \rightarrow A)} \\
\Leftrightarrow & \qquad \{\ \textit{Unfolding}\ \neg\ \} \\
 & A \leftrightarrow \underline{((B \leftrightarrow A) \rightarrow A)} \leftrightarrow \bot \\
\Leftrightarrow & \qquad \{\ \textit{Unfolding}\ \rightarrow\ \} \\
 & \underline{A \leftrightarrow A} \leftrightarrow (B \leftrightarrow A) \vee A \leftrightarrow \bot \\
\Leftrightarrow & \qquad \{\ \textit{Unfolding}\ \top\ \textit{twice}\ \} \\
 & \underline{(B \leftrightarrow A) \vee A} \leftrightarrow \bot \\
\Leftrightarrow & \qquad \{\ \textit{Distributivity of}\ \vee\ \} \\
 & B \vee A \leftrightarrow \underline{A \vee A} \leftrightarrow \bot \\
\Leftrightarrow & \qquad \{\ \textit{Idempotency of}\ \vee\ \} \\
 & B \vee A \leftrightarrow \underline{A} \leftrightarrow \bot \\
\Leftrightarrow & \qquad \{\ P \vee \bot \leftrightarrow P\ \} \\
 & \underline{B \vee A \leftrightarrow \bot \vee A} \leftrightarrow \bot \\
\Leftrightarrow & \qquad \{\ \textit{Distributivity of}\ \vee\ \} \\
 & \underline{(\underline{B \leftrightarrow \bot}) \vee A \leftrightarrow \bot} \\
\Leftrightarrow & \qquad \{\ \textit{Unfolding}\ \neg\ \textit{twice}\ \} \\
 & \underline{\neg(\neg B \vee A)} \\
\Leftrightarrow & \qquad \{\ \neg P \wedge \neg Q \leftrightarrow \neg(P \vee Q)\ \} \\
 & \underline{\neg\neg B} \wedge \neg A \\
\Leftrightarrow & \qquad \{\ \textit{unfolding}\ \neg,\ \textit{unfolding}\ \top\ \} \\
 & B \wedge \neg A
\end{array}
$$

So we can deduce that A is a knave and B is a knight.