Logic
Part II: First-order Logic

Max Schäfer

Formosan Summer School on Logic, Language, and Computation 2010

First-order Logic

一階邏輯

Motivation: First Order Logic

- in mathematics, we want to express propositions about individuals, e.g.

  For every x *we have* $1 + x > x$

- in the example, the individuals are numbers, ranged over by the individual variable x
- we use constants (like 1) and functions (like $+$, arity 2) to construct *terms*
- relations (like $>$, arity 2) can be used to form *atomic propositions* about terms
- atomic propositions are used to construct more complex propositions
- first order logic (FOL) formalises such statements in an abstract setting

Principles of First Order Logic (FOL)

- first order logic formalises reasoning about statements that can refer to individuals through *individual variables*
- a fixed set of function symbols acts on the individuals
- a fixed set of relation symbols expresses predicates on the individuals
- more complex statements can be formed by connectives like $\wedge, \vee, \rightarrow, \neg$ and the quantifiers $\forall, \exists$
- first order logic is sufficient to formalise great parts of mathematics, for example arithmetic

The Language of FOL

- a first order *signature* $\Sigma = \langle \mathtt{F}, \mathcal{R} \rangle$ describes a language with
  - *function letters* $f \in \mathtt{F}$ with arity $\alpha(f) \in \mathbb{N}$
  - *relation letters* $r \in \mathcal{R}$ with arity $\alpha(r) \in \mathbb{N}$
- terms $\mathtt{T}(\Sigma, \mathcal{V})$ over Σ and a set $\mathcal{V}$ of *individual variables* are inductively defined:
  - $\mathcal{V} \subseteq \mathtt{T}(\Sigma, \mathcal{V})$
  - for $f \in \mathtt{F}$ of arity n, $t_1, \ldots, t_n \in \mathtt{T}(\Sigma, \mathcal{V})$, also $f(t_1, \ldots, t_n) \in \mathtt{T}(\Sigma, \mathcal{V})$
- for a 0-ary constant d, we write $d()$ simply as d

Example

Signature $\Sigma_{\mathrm{ar}} = \langle \mathtt{F}_{\mathrm{ar}}, \mathcal{R}_{\mathrm{ar}} \rangle$ of arithmetic:

- $\mathtt{F}_{\mathrm{ar}} = \{\mathbf{0}, s, +, \times\}$, where $\alpha(\mathbf{0}) = 0$, $\alpha(s) = 1$, $\alpha(+) = \alpha(\times) = 2$
- $\mathcal{R}_{\mathrm{ar}} = \{\leq\}$, where $\alpha(\leq) = 2$

- examples for terms from $\mathtt{T}(\Sigma_{\mathrm{ar}}, \{x, y\})$: $\mathbf{0}$, $s(\mathbf{0})$, $s(s(\mathbf{0}))$, $\ldots$, $s(x)$, $\times(s(x), y)$, $s(\times(x, y))$, $\ldots$
- but not $\mathbf{0}(\mathbf{0})$ or $\times(s(\mathbf{0}))$
- $\times(x, y)$ usually written $x \times y$, but still $\times(x, y) \equiv x \times y$

Example

Signature $\Sigma_{\mathrm{ar}} = \langle \mathtt{F}_{\mathrm{ar}}, \mathcal{R}_{\mathrm{ar}} \rangle$ of arithmetic:

- $\mathtt{F}_{\mathrm{ar}} = \{\mathbf{0}, s, +, \times\}$, where $\alpha(\mathbf{0}) = 0$, $\alpha(s) = 1$, $\alpha(+) = \alpha(\times) = 2$
- $\mathcal{R}_{\mathrm{ar}} = \{\leq\}$, where $\alpha(\leq) = 2$

- examples for terms from $\mathtt{T}(\Sigma_{\mathrm{ar}}, \{x, y\})$: $\mathbf{0}$, $s(\mathbf{0})$, $s(s(\mathbf{0}))$, $\ldots$, $s(x)$, $\times(s(x), y)$, $s(\times(x, y))$, $\ldots$
- but not $\mathbf{0}(\mathbf{0})$ or $\times(s(\mathbf{0}))$
- $\times(x, y)$ usually written $x \times y$, but still $\times(x, y) \equiv x \times y$

The Language of FOL (II)

- an *atom* is either
  - of the form $r(t_1, \ldots, t_n)$, where $r \in \mathcal{R}$, $\alpha(r) = n$, $t_1, \ldots, t_n \in \mathtt{T}(\Sigma, \mathcal{V})$
  - or of the form $s \doteq t$, where $s, t \in \mathtt{T}(\Sigma, \mathcal{V})$
- we write r for $r()$ if $\alpha(r) = 0$
- the language of formulas of first-order logic over Σ and $\mathcal{V}$ is given by the following grammar:

  $$\varphi ::= \mathcal{A}(\Sigma, \mathcal{V}) \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \forall \mathcal{V}.\varphi \mid \exists \mathcal{V}.\varphi$$

  where $\mathcal{A}(\Sigma, \mathcal{V})$ are the atoms over Σ and $\mathcal{V}$
- $\forall$ and $\exists$ have the lowest precedence

Example

Atoms over Σ_{ar} and $\mathcal{V} = \{x, y, d, d'\}$:

- $x \doteq y$
- $x + y \doteq y + x$
- $s(s(\mathbf{0})) \times x \leq x + x$

Non-atomic formulas:

- $\neg(x \doteq s(x))$
- $(\exists d.x + d \doteq y) \rightarrow (\exists d'.s(x) + d' \doteq y) \vee s(x) \doteq y$
- $\forall x.x + x \leq x \times x$

Intuitive Semantics of the Quantifiers

- $\forall x.\varphi$ should be understood as "for all values of x, φ holds"
- $\exists x.\varphi$ should be understood as "there is a value of x such that φ holds"
- so the formula

$$\forall x.x \doteq \mathbf{0} \vee \exists y.x \doteq s(y)$$

  could be understood as

  > *every x is either equal to zero, or there exists a number y such that x is its successor*

- **however**, this interpretation relies on an intuitive interpretation of the function symbols s and $\mathbf{0}$ and the relation symbol $\doteq$; it is certainly not true for all interpretations of these symbols!

Intuitive Semantics of the Quantifiers

- $\forall x.\varphi$ should be understood as "for all values of x, φ holds"
- $\exists x.\varphi$ should be understood as "there is a value of x such that φ holds"
- so the formula

$$\forall x.x \doteq \mathbf{0} \vee \exists y.x \doteq s(y)$$

  could be understood as

  > *every x is either equal to zero, or there exists a number y such that x is its successor*

- **however**, this interpretation relies on an intuitive interpretation of the function symbols s and $\mathbf{0}$ and the relation symbol $\doteq$; it is certainly not true for all interpretations of these symbols!

Intuitive Semantics of the Quantifiers

- $\forall x.\varphi$ should be understood as "for all values of x, φ holds"
- $\exists x.\varphi$ should be understood as "there is a value of x such that φ holds"
- so the formula

$$\forall x.x \doteq \mathbf{0} \vee \exists y.x \doteq s(y)$$

  could be understood as

  every x is either equal to zero, or there exists a number y such that x is its successor

- **however**, this interpretation relies on an intuitive interpretation of the function symbols s and $\mathbf{0}$ and the relation symbol $\doteq$; it is certainly not true for all interpretations of these symbols!

Free and Bound Variables

- an appearance of an individual variable is called *bound* if it is within the scope of a quantifier, otherwise it is *free*
  e.g. (free variables are red):

$$x \doteq s(y) \qquad \exists x.x \doteq s(y) \qquad \forall y.\exists x.x \doteq s(y)$$

- the same variable can appear both free and bound:

$$(\forall x.R(x, z) \rightarrow (\exists y.S(y, x))) \wedge T(x)$$

Free and Bound Variables

- an appearance of an individual variable is called *bound* if it is within the scope of a quantifier, otherwise it is *free*
  e.g. (free variables are red):

$$x \doteq s(y) \qquad \exists x.x \doteq s(y) \qquad \forall y.\exists x.x \doteq s(y)$$

- the same variable can appear both free and bound:

$$(\forall x.R(x, z) \rightarrow (\exists y.S(y, x))) \wedge T(x)$$

The Set of Free Variables

- free variables in a term:
  1. $\mathrm{FV}(x) = \{x\}$ for $x \in \mathcal{V}$
  2. $\mathrm{FV}(f(t_1, \ldots, t_n)) = \bigcup_{i \in \{1,\ldots,n\}} \mathrm{FV}(t_i)$
- free variables in a formula:
  1. $\mathrm{FV}(r(t_1, \ldots, t_n)) = \bigcup_{i \in \{1,\ldots,n\}} \mathrm{FV}(t_i)$
  2. $\mathrm{FV}(s \doteq t) = \mathrm{FV}(s) \cup \mathrm{FV}(t)$
  3. $\mathrm{FV}(\bot) = \emptyset$
  4. $\mathrm{FV}(\varphi \wedge \psi) = \mathrm{FV}(\varphi \vee \psi) = \mathrm{FV}(\varphi \rightarrow \psi) = \mathrm{FV}(\varphi) \cup \mathrm{FV}(\psi)$
  5. $\mathrm{FV}(\forall x.\varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$
  6. $\mathrm{FV}(\exists x.\varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$

For example:

- $\mathrm{FV}(x) = \{x\}$, $\mathrm{FV}(\mathbf{0}) = \emptyset$, $\mathrm{FV}(s(y)) = \{y\}$
- $\mathrm{FV}(x \doteq \mathbf{0} \vee x \doteq s(y)) = \{x, y\}$
- $\mathrm{FV}(x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y))) = \{x\}$
- $\mathrm{FV}(\forall x.x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y))) = \emptyset$

The Set of Free Variables

- free variables in a term:
  1. $\mathrm{FV}(x) = \{x\}$ for $x \in \mathcal{V}$
  2. $\mathrm{FV}(f(t_1, \ldots, t_n)) = \bigcup_{i \in \{1,\ldots,n\}} \mathrm{FV}(t_i)$
- free variables in a formula:
  1. $\mathrm{FV}(r(t_1, \ldots, t_n)) = \bigcup_{i \in \{1,\ldots,n\}} \mathrm{FV}(t_i)$
  2. $\mathrm{FV}(s \doteq t) = \mathrm{FV}(s) \cup \mathrm{FV}(t)$
  3. $\mathrm{FV}(\bot) = \emptyset$
  4. $\mathrm{FV}(\varphi \wedge \psi) = \mathrm{FV}(\varphi \vee \psi) = \mathrm{FV}(\varphi \to \psi) = \mathrm{FV}(\varphi) \cup \mathrm{FV}(\psi)$
  5. $\mathrm{FV}(\forall x.\varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$
  6. $\mathrm{FV}(\exists x.\varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$

For example:

- $\mathrm{FV}(x) = \{x\}$, $\mathrm{FV}(\mathbf{0}) = \emptyset$, $\mathrm{FV}(s(y)) = \{y\}$
- $\mathrm{FV}(x \doteq \mathbf{0} \vee x \doteq s(y)) = \{x, y\}$
- $\mathrm{FV}(x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y))) = \{x\}$
- $\mathrm{FV}(\forall x.x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y))) = \emptyset$

The Set of Free Variables

- free variables in a term:
  1. $\mathrm{FV}(x) = \{x\}$ for $x \in \mathcal{V}$
  2. $\mathrm{FV}(f(t_1, \ldots, t_n)) = \bigcup_{i \in \{1,\ldots,n\}} \mathrm{FV}(t_i)$
- free variables in a formula:
  1. $\mathrm{FV}(r(t_1, \ldots, t_n)) = \bigcup_{i \in \{1,\ldots,n\}} \mathrm{FV}(t_i)$
  2. $\mathrm{FV}(s \doteq t) = \mathrm{FV}(s) \cup \mathrm{FV}(t)$
  3. $\mathrm{FV}(\bot) = \emptyset$
  4. $\mathrm{FV}(\varphi \wedge \psi) = \mathrm{FV}(\varphi \vee \psi) = \mathrm{FV}(\varphi \to \psi) = \mathrm{FV}(\varphi) \cup \mathrm{FV}(\psi)$
  5. $\mathrm{FV}(\forall x.\varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$
  6. $\mathrm{FV}(\exists x.\varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$

For example:

- $\mathrm{FV}(x) = \{x\}$, $\mathrm{FV}(\mathbf{0}) = \emptyset$, $\mathrm{FV}(s(y)) = \{y\}$
- $\mathrm{FV}(x \doteq \mathbf{0} \vee x \doteq s(y)) = \{x, y\}$
- $\mathrm{FV}(x \doteq \mathbf{0} \vee (\exists y. x \doteq s(y))) = \{x\}$
- $\mathrm{FV}(\forall x. x \doteq \mathbf{0} \vee (\exists y. x \doteq s(y))) = \emptyset$

The Set of Free Variables

- free variables in a term:
  1. $\mathrm{FV}(x) = \{x\}$ for $x \in \mathcal{V}$
  2. $\mathrm{FV}(f(t_1, \ldots, t_n)) = \bigcup_{i \in \{1,\ldots,n\}} \mathrm{FV}(t_i)$
- free variables in a formula:
  1. $\mathrm{FV}(r(t_1, \ldots, t_n)) = \bigcup_{i \in \{1,\ldots,n\}} \mathrm{FV}(t_i)$
  2. $\mathrm{FV}(s \doteq t) = \mathrm{FV}(s) \cup \mathrm{FV}(t)$
  3. $\mathrm{FV}(\bot) = \emptyset$
  4. $\mathrm{FV}(\varphi \wedge \psi) = \mathrm{FV}(\varphi \vee \psi) = \mathrm{FV}(\varphi \rightarrow \psi) = \mathrm{FV}(\varphi) \cup \mathrm{FV}(\psi)$
  5. $\mathrm{FV}(\forall x.\varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$
  6. $\mathrm{FV}(\exists x.\varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$

For example:

- $\mathrm{FV}(x) = \{x\}$, $\mathrm{FV}(\mathbf{0}) = \emptyset$, $\mathrm{FV}(s(y)) = \{y\}$
- $\mathrm{FV}(x \doteq \mathbf{0} \vee x \doteq s(y)) = \{x, y\}$
- $\mathrm{FV}(x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y))) = \{x\}$
- $\mathrm{FV}(\forall x.x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y))) = \emptyset$

The Set of Free Variables

- free variables in a term:
  1. $\mathrm{FV}(x) = \{x\}$ for $x \in \mathcal{V}$
  2. $\mathrm{FV}(f(t_1, \ldots, t_n)) = \bigcup_{i \in \{1,\ldots,n\}} \mathrm{FV}(t_i)$
- free variables in a formula:
  1. $\mathrm{FV}(r(t_1, \ldots, t_n)) = \bigcup_{i \in \{1,\ldots,n\}} \mathrm{FV}(t_i)$
  2. $\mathrm{FV}(s \doteq t) = \mathrm{FV}(s) \cup \mathrm{FV}(t)$
  3. $\mathrm{FV}(\bot) = \emptyset$
  4. $\mathrm{FV}(\varphi \wedge \psi) = \mathrm{FV}(\varphi \vee \psi) = \mathrm{FV}(\varphi \to \psi) = \mathrm{FV}(\varphi) \cup \mathrm{FV}(\psi)$
  5. $\mathrm{FV}(\forall x.\varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$
  6. $\mathrm{FV}(\exists x.\varphi) = \mathrm{FV}(\varphi) \setminus \{x\}$

For example:

- $\mathrm{FV}(x) = \{x\}$, $\mathrm{FV}(\mathbf{0}) = \emptyset$, $\mathrm{FV}(s(y)) = \{y\}$
- $\mathrm{FV}(x \doteq \mathbf{0} \vee x \doteq s(y)) = \{x, y\}$
- $\mathrm{FV}(x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y))) = \{x\}$
- $\mathrm{FV}(\forall x.x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y))) = \emptyset$

Substitution in Terms and Formulas

- substituting a term t for a variable x in a term s ($s[t/x]$):
  1. $y[t/x] = \begin{cases} t & \text{if } x = y, \\ y & \text{otherwise} \end{cases}$
  2. $f(t_1, \ldots, t_n)[t/x] = f(t_1[t/x], \ldots, t_n[t/x])$
- on formulas, the definition is
  1. $r(t_1, \ldots, t_n)[t/x] = r(t_1[t/x], \ldots, t_n[t/x])$
  2. $(s_1 \doteq s_2)[t/x] = (s_1[t/x] \doteq s_2[t/x])$
  3. $\bot[t/x] = \bot$
  4. $(\varphi \circ \psi)[t/x] = (\varphi[t/x] \circ \psi[t/x])$, where $\circ$ is either $\wedge$ or $\vee$ or $\rightarrow$
  5. $(\forall y.\varphi)[t/x] = \begin{cases} \forall y.\varphi & \text{if } x = y, \\ \forall y.(\varphi[t/x]) & \text{if } x \neq y,\ y \notin \mathrm{FV}(t) \end{cases}$

     $(\exists y.\varphi)[t/x] = \begin{cases} \exists y.\varphi & \text{if } x = y, \\ \exists y.(\varphi[t/x]) & \text{if } x \neq y,\ y \notin \mathrm{FV}(t) \end{cases}$

Example

- $x[s(\mathbf{0})/x] \equiv s(\mathbf{0})$
- $y[s(\mathbf{0})/x] \equiv y$
- $(x \doteq \mathbf{0} \vee x \doteq s(y))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \vee s(\mathbf{0}) \doteq s(y)$
- $(x \doteq \mathbf{0} \vee (\exists y. x \doteq s(y)))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \vee (\exists y. s(\mathbf{0}) \doteq s(y))$
- $(x \doteq \mathbf{0} \vee x \doteq s(y))[s(y)/x] \equiv s(y) \doteq \mathbf{0} \vee s(y) \doteq s(y)$
- $(x \doteq \mathbf{0} \vee (\exists y. x \doteq s(y)))[s(y)/x]$ is not defined
- $(\forall x. x \doteq \mathbf{0} \vee (\exists y. x \doteq s(y)))[s(\mathbf{0})/x]$
  $\equiv (\forall x. x \doteq \mathbf{0} \vee (\exists y. x \doteq s(y)))$

Example

- $x[s(\mathbf{0})/x] \equiv s(\mathbf{0})$
- $y[s(\mathbf{0})/x] \equiv y$
- $(x \doteq \mathbf{0} \vee x \doteq s(y))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \vee s(\mathbf{0}) \doteq s(y)$
- $(x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \vee (\exists y.s(\mathbf{0}) \doteq s(y))$
- $(x \doteq \mathbf{0} \vee x \doteq s(y))[s(y)/x] \equiv s(y) \doteq \mathbf{0} \vee s(y) \doteq s(y)$
- $(x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))[s(y)/x]$ is not defined
- $(\forall x.x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))[s(\mathbf{0})/x]$
  $\equiv (\forall x.x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))$

Example

- $x[s(\mathbf{0})/x] \equiv s(\mathbf{0})$
- $y[s(\mathbf{0})/x] \equiv y$
- $(x \doteq \mathbf{0} \vee x \doteq s(y))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \vee s(\mathbf{0}) \doteq s(y)$
- $(x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \vee (\exists y.s(\mathbf{0}) \doteq s(y))$
- $(x \doteq \mathbf{0} \vee x \doteq s(y))[s(y)/x] \equiv s(y) \doteq \mathbf{0} \vee s(y) \doteq s(y)$
- $(x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))[s(y)/x]$ is not defined
- $(\forall x.x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))[s(\mathbf{0})/x]$
  $\equiv (\forall x.x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))$

Example

- $x[s(\mathbf{0})/x] \equiv s(\mathbf{0})$
- $y[s(\mathbf{0})/x] \equiv y$
- $(x \doteq \mathbf{0} \lor x \doteq s(y))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \lor s(\mathbf{0}) \doteq s(y)$
- $(x \doteq \mathbf{0} \lor (\exists y.x \doteq s(y)))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \lor (\exists y.s(\mathbf{0}) \doteq s(y))$
- $(x \doteq \mathbf{0} \lor x \doteq s(y))[s(y)/x] \equiv s(y) \doteq \mathbf{0} \lor s(y) \doteq s(y)$
- $(x \doteq \mathbf{0} \lor (\exists y.x \doteq s(y)))[s(y)/x]$ is not defined
- $(\forall x.x \doteq \mathbf{0} \lor (\exists y.x \doteq s(y)))[s(\mathbf{0})/x]$
  $\equiv (\forall x.x \doteq \mathbf{0} \lor (\exists y.x \doteq s(y)))$

Example

- $x[s(\mathbf{0})/x] \equiv s(\mathbf{0})$
- $y[s(\mathbf{0})/x] \equiv y$
- $(x \doteq \mathbf{0} \vee x \doteq s(y))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \vee s(\mathbf{0}) \doteq s(y)$
- $(x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \vee (\exists y.s(\mathbf{0}) \doteq s(y))$
- $(x \doteq \mathbf{0} \vee x \doteq s(y))[s(y)/x] \equiv s(y) \doteq \mathbf{0} \vee s(y) \doteq s(y)$
- $(x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))[s(y)/x]$ is not defined
- $(\forall x.x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))[s(\mathbf{0})/x]$
  $\equiv (\forall x.x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))$

Example

- $x[s(\mathbf{0})/x] \equiv s(\mathbf{0})$
- $y[s(\mathbf{0})/x] \equiv y$
- $(x \doteq \mathbf{0} \lor x \doteq s(y))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \lor s(\mathbf{0}) \doteq s(y)$
- $(x \doteq \mathbf{0} \lor (\exists y.x \doteq s(y)))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \lor (\exists y.s(\mathbf{0}) \doteq s(y))$
- $(x \doteq \mathbf{0} \lor x \doteq s(y))[s(y)/x] \equiv s(y) \doteq \mathbf{0} \lor s(y) \doteq s(y)$
- $(x \doteq \mathbf{0} \lor (\exists y.x \doteq s(y)))[s(y)/x]$ is not defined
- $(\forall x.x \doteq \mathbf{0} \lor (\exists y.x \doteq s(y)))[s(\mathbf{0})/x]$
  $\equiv (\forall x.x \doteq \mathbf{0} \lor (\exists y.x \doteq s(y)))$

Example

- $x[s(\mathbf{0})/x] \equiv s(\mathbf{0})$
- $y[s(\mathbf{0})/x] \equiv y$
- $(x \doteq \mathbf{0} \vee x \doteq s(y))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \vee s(\mathbf{0}) \doteq s(y)$
- $(x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))[s(\mathbf{0})/x] \equiv s(\mathbf{0}) \doteq \mathbf{0} \vee (\exists y.s(\mathbf{0}) \doteq s(y))$
- $(x \doteq \mathbf{0} \vee x \doteq s(y))[s(y)/x] \equiv s(y) \doteq \mathbf{0} \vee s(y) \doteq s(y)$
- $(x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))[s(y)/x]$ is not defined
- $(\forall x.x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))[s(\mathbf{0})/x]$
  $\equiv (\forall x.x \doteq \mathbf{0} \vee (\exists y.x \doteq s(y)))$

Alpha Equivalence

- $\forall x.\varphi$ *alpha reduces* to $\forall y.\varphi'$ if $\varphi' \equiv \varphi[y/x]$
  $\exists x.\varphi$ *alpha reduces* to $\exists y.\varphi'$ if $\varphi' \equiv \varphi[y/x]$
- φ is called *alpha equivalent* to ψ (written $\varphi \equiv_\alpha \psi$), if ψ results from φ by any number of alpha reductions on subformulas of φ
- Examples:
  - $(\forall x.R(x,x)) \equiv_\alpha (\forall y.R(y,y))$
  - $(\forall x.\exists x.S(x)) \equiv_\alpha (\forall y.\exists x.S(x)) \equiv_\alpha (\forall y.\exists z.S(z))$
  - $(\forall x.\exists y.T(x,y)) \not\equiv_\alpha (\forall x.\exists x.T(x,x))$

Notice that alpha reduction *never* changes the names of free variables.

Alpha Equivalence

- $\forall x.\varphi$ *alpha reduces* to $\forall y.\varphi'$ if $\varphi' \equiv \varphi[y/x]$
  $\exists x.\varphi$ *alpha reduces* to $\exists y.\varphi'$ if $\varphi' \equiv \varphi[y/x]$
- φ is called *alpha equivalent* to ψ (written $\varphi \equiv_\alpha \psi$), if ψ results from φ by any number of alpha reductions on subformulas of φ
- Examples:
  - $(\forall x.R(x,x)) \equiv_\alpha (\forall y.R(y,y))$
  - $(\forall x.\exists x.S(x)) \equiv_\alpha (\forall y.\exists x.S(x)) \equiv_\alpha (\forall y.\exists z.S(z))$
  - $(\forall x.\exists y.T(x,y)) \not\equiv_\alpha (\forall x.\exists x.T(x,x))$

Notice that alpha reduction *never* changes the names of free variables.

Alpha Equivalence

- $\forall x.\varphi$ *alpha reduces* to $\forall y.\varphi'$ if $\varphi' \equiv \varphi[y/x]$
  $\exists x.\varphi$ *alpha reduces* to $\exists y.\varphi'$ if $\varphi' \equiv \varphi[y/x]$
- φ is called *alpha equivalent* to ψ (written $\varphi \equiv_\alpha \psi$), if ψ results from φ by any number of alpha reductions on subformulas of φ
- Examples:
  - $(\forall x.R(x, x)) \equiv_\alpha (\forall y.R(y, y))$
  - $(\forall x.\exists x.S(x)) \equiv_\alpha (\forall y.\exists x.S(x)) \equiv_\alpha (\forall y.\exists z.S(z))$
  - $(\forall x.\exists y.T(x, y)) \not\equiv_\alpha (\forall x.\exists x.T(x, x))$

Notice that alpha reduction *never* changes the names of free variables.

Alpha Equivalence

- $\forall x.\varphi$ *alpha reduces* to $\forall y.\varphi'$ if $\varphi' \equiv \varphi[y/x]$
  $\exists x.\varphi$ *alpha reduces* to $\exists y.\varphi'$ if $\varphi' \equiv \varphi[y/x]$
- φ is called *alpha equivalent* to ψ (written $\varphi \equiv_\alpha \psi$), if ψ results from φ by any number of alpha reductions on subformulas of φ
- Examples:
  - $(\forall x.R(x,x)) \equiv_\alpha (\forall y.R(y,y))$
  - $(\forall x.\exists x.S(x)) \equiv_\alpha (\forall y.\exists x.S(x)) \equiv_\alpha (\forall y.\exists z.S(z))$
  - $(\forall x.\exists y.T(x,y)) \not\equiv_\alpha (\forall x.\exists x.T(x,x))$

Notice that alpha reduction *never* changes the names of free variables.

Alpha Equivalence

- $\forall x.\varphi$ *alpha reduces* to $\forall y.\varphi'$ if $\varphi' \equiv \varphi[y/x]$
  $\exists x.\varphi$ *alpha reduces* to $\exists y.\varphi'$ if $\varphi' \equiv \varphi[y/x]$
- φ is called *alpha equivalent* to ψ (written $\varphi \equiv_\alpha \psi$), if ψ results from φ by any number of alpha reductions on subformulas of φ
- Examples:
  - $(\forall x.R(x,x)) \equiv_\alpha (\forall y.R(y,y))$
  - $(\forall x.\exists x.S(x)) \equiv_\alpha (\forall y.\exists x.S(x)) \equiv_\alpha (\forall y.\exists z.S(z))$
  - $(\forall x.\exists y.T(x,y)) \not\equiv_\alpha (\forall x.\exists x.T(x,x))$

Notice that alpha reduction *never* changes the names of free variables.

Renaming Away

- we do not distinguish between alpha equivalent formulas
- hence, we can use alpha reduction to rename problematic bound variables such that substitution is always defined
- example:

$$(x \doteq \mathbf{0} \vee \exists y.x \doteq s(y))[s(y)/x]$$

is not defined, but

$$x \doteq \mathbf{0} \vee \exists y.x \doteq s(y) \equiv_\alpha x \doteq \mathbf{0} \vee \exists z.x \doteq s(z)$$

thus we can define

$$\begin{aligned}(x \doteq \mathbf{0} \vee \exists y.x \doteq s(y))[s(y)/x] \quad &:= (x \doteq \mathbf{0} \vee \exists z.x \doteq s(z))[s(y)/x] \\ &\equiv s(y) \doteq \mathbf{0} \vee \exists z.s(y) \doteq s(z)\end{aligned}$$

Renaming Away

- we do not distinguish between alpha equivalent formulas
- hence, we can use alpha reduction to rename problematic bound variables such that substitution is always defined
- example:

$$(x \doteq \mathbf{0} \vee \exists y.x \doteq s(y))[s(y)/x]$$

is not defined, but

$$x \doteq \mathbf{0} \vee \exists y.x \doteq s(y) \equiv_\alpha x \doteq \mathbf{0} \vee \exists z.x \doteq s(z)$$

thus we can define

$$\begin{aligned} (x \doteq \mathbf{0} \vee \exists y.x \doteq s(y))[s(y)/x] \quad &:= (x \doteq \mathbf{0} \vee \exists z.x \doteq s(z))[s(y)/x] \\ &\equiv s(y) \doteq \mathbf{0} \vee \exists z.s(y) \doteq s(z) \end{aligned}$$

Renaming Away

- we do not distinguish between alpha equivalent formulas
- hence, we can use alpha reduction to rename problematic bound variables such that substitution is always defined
- example:

$$(x \doteq \mathbf{0} \vee \exists y.x \doteq s(y))[s(y)/x]$$

is not defined, but

$$x \doteq \mathbf{0} \vee \exists y.x \doteq s(y) \equiv_\alpha x \doteq \mathbf{0} \vee \exists z.x \doteq s(z)$$

thus we can define

$$\begin{aligned}(x \doteq \mathbf{0} \vee \exists y.x \doteq s(y))[s(y)/x] \quad &:= (x \doteq \mathbf{0} \vee \exists z.x \doteq s(z))[s(y)/x] \\ &\equiv s(y) \doteq \mathbf{0} \vee \exists z.s(y) \doteq s(z)\end{aligned}$$

Renaming Away

- we do not distinguish between alpha equivalent formulas
- hence, we can use alpha reduction to rename problematic bound variables such that substitution is always defined
- example:

$$(x \doteq \mathbf{0} \lor \exists y.x \doteq s(y))[s(y)/x]$$

is not defined, but

$$x \doteq \mathbf{0} \lor \exists y.x \doteq s(y) \equiv_\alpha x \doteq \mathbf{0} \lor \exists z.x \doteq s(z)$$

thus we can define

$$\begin{aligned}(x \doteq \mathbf{0} \lor \exists y.x \doteq s(y))[s(y)/x] \quad &:= (x \doteq \mathbf{0} \lor \exists z.x \doteq s(z))[s(y)/x] \\ &\equiv s(y) \doteq \mathbf{0} \lor \exists z.s(y) \doteq s(z)\end{aligned}$$

Motivation: Semantics of FOL

- the function and relation symbols in Σ have no predefined meaning
- thus, we do not know if $\forall x. x \doteq \mathbf{0}$ is true
- but some sentences are true no matter how the symbols are interpreted, e.g.:

$$(\forall x. \forall y. R(x, y) \rightarrow R(y, x)) \rightarrow R(a, b) \rightarrow R(b, a)$$

- how do we evaluate, e.g., $\forall x. \neg R(x, x)$?
  - we need to know what x can stand for, and for which of these values R is true
  - then we would like to evaluate $\neg R(x, x)$, where x is bound to any of its possible values
- thus, we need to consider not only the interpretation of the function and relation symbols, but also variable bindings

Motivation: Semantics of FOL

- the function and relation symbols in Σ have no predefined meaning
- thus, we do not know if $\forall x . x \doteq \mathbf{0}$ is true
- but some sentences are true no matter how the symbols are interpreted, e.g.:

$$(\forall x . \forall y . R(x, y) \rightarrow R(y, x)) \rightarrow R(a, b) \rightarrow R(b, a)$$

- how do we evaluate, e.g., $\forall x . \neg R(x, x)$?
  - we need to know what x can stand for, and for which of these values R is true
  - then we would like to evaluate $\neg R(x, x)$, where x is bound to any of its possible values
- thus, we need to consider not only the interpretation of the function and relation symbols, but also variable bindings

Motivation: Semantics of FOL

- the function and relation symbols in Σ have no predefined meaning
- thus, we do not know if $\forall x.x \doteq \mathbf{0}$ is true
- but some sentences are true no matter how the symbols are interpreted, e.g.:

$$(\forall x.\forall y.R(x,y) \rightarrow R(y,x)) \rightarrow R(a,b) \rightarrow R(b,a)$$

- how do we evaluate, e.g., $\forall x.\neg R(x,x)$?
  - we need to know what x can stand for, and for which of these values R is true
  - then we would like to evaluate $\neg R(x,x)$, where x is bound to any of its possible values
- thus, we need to consider not only the interpretation of the function and relation symbols, but also variable bindings

Semantics: Structures, Interpretations and Assignments

- a (first order) structure $\mathcal{M} = \langle D, I \rangle$ for a signature Σ consists of
  - a non-empty set D, the *domain*
  - an interpretation $I = \langle [\![\,]\!]_{\mathtt{F}}, [\![\,]\!]_{\mathcal{R}} \rangle$ such that
    - for every $f \in \mathtt{F}$ with $\alpha(f) = n$, $[\![f]\!]_{\mathtt{F}} \colon D^n \to D$
    - for every $r \in \mathcal{R}$ with $\alpha(r) = n$, $[\![r]\!]_{\mathcal{R}} \colon D^n \to \mathcal{B}$
- a *variable assignment* on I is a function $\sigma \colon \mathcal{V} \to D$
  We write $\sigma[x := t]$ for the assignment

$$y \mapsto \begin{cases} t & \text{if } x = y \\ \sigma(y) & \text{otherwise} \end{cases}$$

Semantics: Interpreting Terms and Formulas

- interpretation of terms over $\mathcal{M}$ and σ:
  - $[\![x]\!]_{\mathcal{M},\sigma} := \sigma(x)$
  - $[\![f(t_1,\ldots,t_n)]\!]_{\mathcal{M},\sigma} := [\![f]\!]_{\mathtt{F}}([\![t_1]\!]_{\mathcal{M},\sigma},\ldots,[\![t_n]\!]_{\mathcal{M},\sigma})$
- interpretation of formulas:
  - $[\![r(t_1,\ldots,t_n)]\!]_{\mathcal{M},\sigma} := [\![r]\!]_{\mathcal{R}}([\![t_1]\!]_{\mathcal{M},\sigma},\ldots,[\![t_n]\!]_{\mathcal{M},\sigma})$
  - $[\![s \doteq t]\!]_{\mathcal{M},\sigma} := \mathtt{T}$ if $[\![s]\!]_{\mathcal{M},\sigma} = [\![t]\!]_{\mathcal{M},\sigma}$, otherwise $[\![s \doteq t]\!]_{\mathcal{M},\sigma} := \mathtt{F}$
  - $[\![\bot]\!]_{\mathcal{M},\sigma}$, $[\![\varphi \wedge \psi]\!]_{\mathcal{M},\sigma}$, etc.: as before
  - $[\![\forall x.\varphi]\!]_{\mathcal{M},\sigma} := \begin{cases} \mathtt{T} & \text{if, for all } d \in D,\ [\![\varphi]\!]_{\mathcal{M},\sigma[x:=d]} = \mathtt{T}, \\ \mathtt{F} & \text{otherwise} \end{cases}$
  - $[\![\exists x.\varphi]\!]_{\mathcal{M},\sigma} := \begin{cases} \mathtt{T} & \text{if there is } d \in D \text{ with } [\![\varphi]\!]_{\mathcal{M},\sigma[x:=d]} = \mathtt{T}, \\ \mathtt{F} & \text{otherwise} \end{cases}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\,]\!]_{\mathtt{F}}, [\![\,]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x.\forall y.\neg(x \doteq y) \wedge x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\ ]\!]_{\mathtt{F}}, [\![\ ]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x. \forall y. \neg(x \doteq y) \wedge x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\,]\!]_{\mathtt{F}}, [\![\,]\!]_{\mathcal{R}} \rangle\rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x.\forall y.\neg(x \doteq y) \wedge x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\,]\!]_{\mathtt{F}}, [\![\,]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x.\forall y.\neg(x \doteq y) \land x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\ ]\!]_{\mathtt{F}}, [\![\ ]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x.\forall y.\neg(x \doteq y) \land x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\,]\!]_{\mathtt{F}}, [\![\,]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x.\forall y.\neg(x \doteq y) \land x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\,]\!]_{\mathtt{F}}, [\![\,]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x.\forall y.\neg(x \doteq y) \wedge x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\,]\!]_{\mathtt{F}}, [\![\,]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x.\forall y.\neg(x \doteq y) \land x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\,]\!]_{\mathtt{F}}, [\![\,]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x. \forall y. \neg(x \doteq y) \wedge x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\,]\!]_{\mathtt{F}}, [\![\,]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x. \forall y. \neg(x \doteq y) \wedge x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\,]\!]_{\mathtt{F}}, [\![\,]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x.\forall y.\neg(x \doteq y) \wedge x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\,]\!]_{\mathtt{F}}, [\![\,]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x.\forall y.\neg(x \doteq y) \wedge x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\ ]\!]_{\mathtt{F}}, [\![\ ]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x.\forall y.\neg(x \doteq y) \wedge x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Example

A structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![\,]\!]_{\mathtt{F}}, [\![\,]\!]_{\mathcal{R}} \rangle \rangle$ for Σ_{ar}

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$
- $[\![\leq]\!]_{\mathcal{R}}(m, n) = \begin{cases} \mathtt{T} & \text{if } m \leq n \\ \mathtt{F} & \text{otherwise} \end{cases}$

Consider $\sigma = \{x \mapsto 0, y \mapsto 1\}$, then

- $[\![x]\!]_{\mathcal{M},\sigma} = 0$, $[\![y + y]\!]_{\mathcal{M},\sigma} = 2$, $[\![\mathbf{0}]\!]_{\mathcal{M},\sigma} = 0$, $[\![s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = 2$, $[\![y \doteq s(s(\mathbf{0}))]\!]_{\mathcal{M},\sigma} = \mathtt{F}$
- $[\![\bot]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, $[\![\neg(x \doteq y)]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $[\![\exists x. \forall y. \neg(x \doteq y) \land x \leq y]\!]_{\mathcal{M},\sigma} = \mathtt{F}$

Satisfiability and Validity

For a structure $\mathcal{M}$, an assignment σ, a formula φ and a set Γ of formulas, we define

- $\mathcal{M}, \sigma \models \varphi$: $[\![\varphi]\!]_{\mathcal{M},\sigma} = \mathtt{T}$
- $\mathcal{M} \models \varphi$ ("M is a model for φ"): $\mathcal{M}, \sigma \models \varphi$ for any σ
- $\varphi \Rightarrow \psi$: if $\mathcal{M}, \sigma \models \varphi$ then also $\mathcal{M}, \sigma \models \psi$
- $\models \varphi$ ("φ is valid"): $\mathcal{M} \models \varphi$ for any structure $\mathcal{M}$
- $\mathcal{M}, \sigma \models \Gamma$: for all $\gamma \in \Gamma$ we have $\mathcal{M}, \sigma \models \gamma$
- $\Gamma \models \varphi$: for any $\mathcal{M}$ and σ with $\mathcal{M}, \sigma \models \Gamma$ we also have $\mathcal{M}, \sigma \models \varphi$

Example: $\models \exists x.D(x) \rightarrow (\forall y.D(y))$ ("Drinker Paradox")

Proof of $\models \exists x.D(x) \to (\forall y.D(y))$

Take the signature $\Sigma_D = \langle \emptyset, \{D\} \rangle$ with $\alpha(D) = 1$ and the set $\mathcal{V}_D = \{x, y\}$ of variables; the Drinker Paradox is clearly a formula over Σ_D and $\mathcal{V}_D$.

Proof of $\models \exists x.D(x) \rightarrow (\forall y.D(y))$

- $\Sigma_D = \langle \emptyset, \{D/1\} \rangle$, $\mathcal{V}_D = \{x, y\}$

Now assume we are given an arbitrary structure $\mathcal{M} = \langle E, \langle [\![\,]\!]_{\mathrm{F}}, [\![\,]\!]_{\mathcal{R}} \rangle \rangle$ and a variable assignment $\sigma \colon \mathcal{V}_D \rightarrow E$. By our definition of semantics, E is a non-empty set; pick an element $e_0 \in E$.

Proof of $\models \exists x.D(x) \rightarrow (\forall y.D(y))$

- $\Sigma_D = \langle \emptyset, \{D/1\} \rangle$, $\mathcal{V}_D = \{x, y\}$
- structure $\mathcal{M}$, domain E, assignment σ; choose $e_o \in E$

Proof of $\models \exists x.D(x) \to (\forall y.D(y))$

- $\Sigma_D = \langle \emptyset, \{D/1\} \rangle$, $\mathcal{V}_D = \{x, y\}$
- structure $\mathcal{M}$, domain E, assignment σ; choose $e_o \in E$

Observe that $[\![D]\!]_{\mathcal{R}}$ is a function from E to $\mathcal{B}$, i.e. $[\![D]\!]_{\mathcal{R}}(e)$ is either $\mathtt{T}$ or $\mathtt{F}$ for every $e \in E$.

Proof of $\models \exists x.D(x) \to (\forall y.D(y))$

- $\Sigma_D = \langle \emptyset, \{D/1\} \rangle$, $\mathcal{V}_D = \{x, y\}$
- structure $\mathcal{M}$, domain E, assignment σ; choose $e_o \in E$
- $[\![D]\!]_{\mathcal{R}} \colon E \to \mathcal{B}$

Proof of $\models \exists x.D(x) \to (\forall y.D(y))$

- $\Sigma_D = \langle \emptyset, \{D/1\} \rangle$, $\mathcal{V}_D = \{x, y\}$
- structure $\mathcal{M}$, domain E, assignment σ; choose $e_o \in E$
- $[\![D]\!]_{\mathcal{R}}: E \to \mathcal{B}$
- We now distinguish two cases:

Proof of $\models \exists x.D(x) \to (\forall y.D(y))$

- $\Sigma_D = \langle \emptyset, \{D/1\} \rangle$, $\mathcal{V}_D = \{x, y\}$
- structure $\mathcal{M}$, domain E, assignment σ; choose $e_o \in E$
- $[\![D]\!]_\mathcal{R} \colon E \to \mathcal{B}$
- We now distinguish two cases:
  - If $[\![D]\!]_\mathcal{R}(e) = \mathtt{T}$ for all $e \in E$, then

    $$[\![D(y)]\!]_{\mathcal{M},\sigma[x:=e_0][y:=e]} = [\![D]\!]_\mathcal{R}(e) = \mathtt{T}$$

    for all $e \in E$, hence

    $$[\![\forall y.D(y)]\!]_{\mathcal{M},\sigma[x:=e_0]} = \mathtt{T}$$

    Certainly also

    $$[\![D(x)]\!]_{\mathcal{M},\sigma[x:=e_0]} = \mathtt{T}$$

    and thus

    $$[\![D(x) \to (\forall y.D(y))]\!]_{\mathcal{M},\sigma[x:=e_0]} = \mathtt{T}$$

    This shows that

    $$[\![\exists x.D(x) \to (\forall y.D(y))]\!]_{\mathcal{M},\sigma} = \mathtt{T}.$$

Proof of $\models \exists x.D(x) \to (\forall y.D(y))$

- $\Sigma_D = \langle \emptyset, \{D/1\} \rangle$, $\mathcal{V}_D = \{x, y\}$
- structure $\mathcal{M}$, domain E, assignment σ; choose $e_o \in E$
- $[\![D]\!]_{\mathcal{R}} \colon E \to \mathcal{B}$
- We now distinguish two cases:
  - $[\![D]\!]_{\mathcal{R}}(e) = \mathtt{T}$ for all e: $[\![\exists x.D(x) \to (\forall y.D(y))]\!]_{\mathcal{M},\sigma} = \mathtt{T}$.
  - Otherwise, $[\![D]\!]_{\mathcal{R}}(e_1)$ is $\mathtt{F}$ for some e_1, hence

    $$[\![D(x)]\!]_{\mathcal{M},\sigma[x:=e_1]} = \mathtt{F}$$

    But then,

    $$[\![D(x) \to (\forall y.D(y))]\!]_{\mathcal{M},\sigma[x:=e_1]} = \mathtt{T}$$

    and consequently

    $$[\![\exists x.D(x) \to (\forall y.D(y))]\!]_{\mathcal{M},\sigma} = \mathtt{T}.$$

Proof of $\models \exists x.D(x) \rightarrow (\forall y.D(y))$

- $\Sigma_D = \langle \emptyset, \{D/1\} \rangle$, $\mathcal{V}_D = \{x, y\}$
- structure $\mathcal{M}$, domain E, assignment σ; choose $e_o \in E$
- $[\![D]\!]_\mathcal{R} \colon E \rightarrow \mathcal{B}$
- We now distinguish two cases:
  - $[\![D]\!]_\mathcal{R}(e) = \mathtt{T}$ for all e: $[\![\exists x.D(x) \rightarrow (\forall y.D(y))]\!]_{\mathcal{M},\sigma} = \mathtt{T}$.
  - Otherwise, too.

In conclusion, we have shown that $\models_{\mathcal{M},\sigma} \exists x.D(x) \rightarrow (\forall y.D(y))$ for arbitrary $\mathcal{M}$ and σ, thus establishing

$$\models \exists x.D(x) \rightarrow (\forall y.D(y)).$$

Basic Results

Fix some signature Σ and a set $\mathcal{V}$ of variables.

Lemma (agreement lemma)

Let $\mathcal{M}$ be a structure for Σ, φ a formula, and σ, σ' variable assignments such that $\sigma(x) = \sigma'(x)$ for all $x \in \mathrm{FV}(\varphi)$. Then $\mathcal{M}, \sigma \models \varphi$ iff $\mathcal{M}, \sigma' \models \varphi$.

Corollary

The interpretation of a closed formula is independent of variable assignments.

Lemma (alpha equivalent formulas are semantically equivalent)

Alpha equivalent formulas evaluate to the same truth value.

Basic Results

Fix some signature Σ and a set $\mathcal{V}$ of variables.

Lemma (agreement lemma)

Let $\mathcal{M}$ be a structure for Σ, φ a formula, and σ, σ' variable assignments such that $\sigma(x) = \sigma'(x)$ for all $x \in \mathrm{FV}(\varphi)$. Then $\mathcal{M}, \sigma \models \varphi$ iff $\mathcal{M}, \sigma' \models \varphi$.

Corollary

The interpretation of a closed formula is independent of variable assignments.

Lemma (alpha equivalent formulas are semantically equivalent)

Alpha equivalent formulas evaluate to the same truth value.

Basic Results

Fix some signature Σ and a set $\mathcal{V}$ of variables.

Lemma (agreement lemma)

Let $\mathcal{M}$ be a structure for Σ, φ a formula, and σ, σ' variable assignments such that $\sigma(x) = \sigma'(x)$ for all $x \in \mathrm{FV}(\varphi)$. Then $\mathcal{M}, \sigma \models \varphi$ iff $\mathcal{M}, \sigma' \models \varphi$.

Corollary

The interpretation of a closed formula is independent of variable assignments.

Lemma (alpha equivalent formulas are semantically equivalent)

Alpha equivalent formulas evaluate to the same truth value.

Some Equivalences of FOL

- $(\forall x.\varphi) \Leftrightarrow \neg(\exists x.\neg\varphi)$
- $(\forall x.\varphi \wedge \psi) \Leftrightarrow (\forall x.\varphi) \wedge (\forall x.\psi)$
- $(\exists x.\varphi \vee \psi) \Leftrightarrow (\exists x.\varphi) \vee (\exists x.\psi)$
- $(\forall x.\forall y.\varphi) \Leftrightarrow (\forall y.\forall x.\varphi)$
- $(\exists x.\exists y.\varphi) \Leftrightarrow (\exists y.\exists x.\varphi)$
- $(\exists x.\forall y.\varphi) \rightarrow (\forall y.\exists x.\varphi)$, but *not* vice versa

Truth Tables for FOL?

- for $\varphi \in \mathtt{PF}$, we can always find out whether $\models \varphi$ by drawing a truth table
- how about $\varphi \in \mathtt{FOL}$?
  - we need to consider all possible structures
  - in particular, all possible domains, all possible functions over them
  - but domains could be infinite...
- unfortunate truth:

Theorem (Undecidability of First Order Logic)

Given an arbitrary first order formula φ, it is undecidable whether $\models \varphi$.

Sequent Calculus

相繼式演算

Principles of Sequent Calculus

- sequent: $\Gamma \vdash \Delta$

  where Γ and Δ are finite sets of formulas
- intuitive meaning: if every formula in Γ is true, then one formula in Δ is true
- sequent calculus LK: system of derivation rules
- allows us to derive sequents that are correct according to the intuitive interpretation

Example Derivation

basic sequent → $P, Q \vdash P$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P, Q \vdash P \qquad P, Q \vdash Q}{P, Q \vdash P \land Q}\,(\land R)}{P \vdash P \land Q, \neg Q}\,(\neg R)}{\vdash P \land Q, \neg P, \neg Q}\,(\neg R)}{\neg(P \land Q) \vdash \neg P, \neg Q}\,(\neg L)}{\neg(P \land Q) \vdash \neg P \lor \neg Q}\,(\lor R)
$$

Example Derivation

basic sequent

derivation step

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P, Q \vdash P \qquad P, Q \vdash Q}{P, Q \vdash P \land Q}\ (\land R)}{P \vdash P \land Q, \neg Q}\ (\neg R)}{\vdash P \land Q, \neg P, \neg Q}\ (\neg R)}{\neg(P \land Q) \vdash \neg P, \neg Q}\ (\neg L)}{\neg(P \land Q) \vdash \neg P \lor \neg Q}\ (\lor R)
$$

Example Derivation

basic sequent

antecedents

derivation step

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P, Q \vdash P \qquad P, Q \vdash Q}{P, Q \vdash P \land Q}\,(\land\mathrm{R})}{P \vdash P \land Q, \neg Q}\,(\neg\mathrm{R})}{\vdash P \land Q, \neg P, \neg Q}\,(\neg\mathrm{R})}{\neg(P \land Q) \vdash \neg P, \neg Q}\,(\neg\mathrm{L})}{\neg(P \land Q) \vdash \neg P \lor \neg Q}\,(\lor\mathrm{R})
$$

Example Derivation

$$
\cfrac{\cfrac{\cfrac{\cfrac{P, Q \vdash P \qquad P, Q \vdash Q}{P, Q \vdash P \wedge Q}\ (\wedge\mathrm{R})}{P \vdash P \wedge Q, \neg Q}\ (\neg\mathrm{R})}{\vdash P \wedge Q, \neg P, \neg Q}\ (\neg\mathrm{R})}{\cfrac{\neg(P \wedge Q) \vdash \neg P, \neg Q}{\neg(P \wedge Q) \vdash \neg P \vee \neg Q}\ (\vee\mathrm{R})}\ (\neg\mathrm{L})
$$

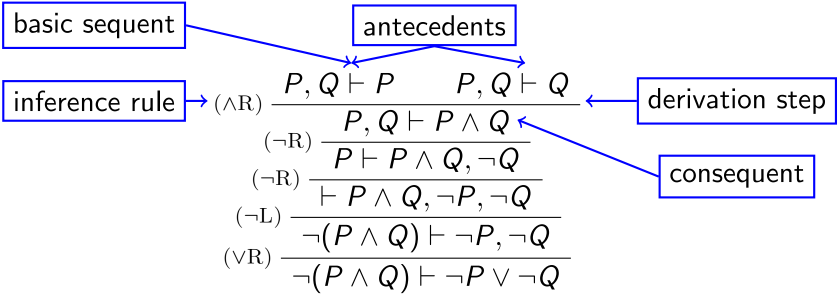
basic sequent

antecedents

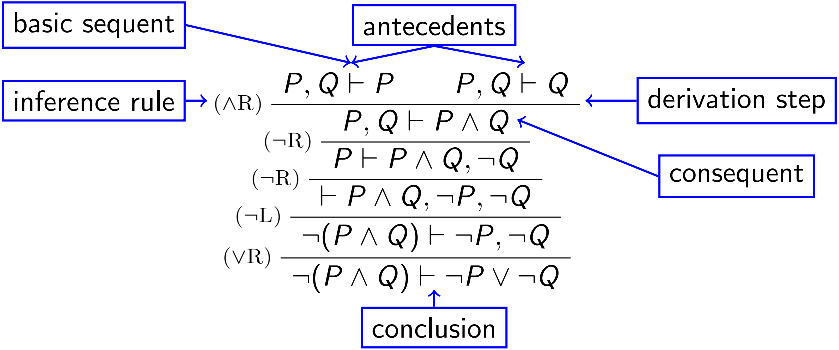
derivation step

consequent

Example Derivation



Example Derivation



Example Derivation (ctd.)

$$\neg(P \wedge Q) \vdash \neg P \vee \neg Q$$

$$(\neg\mathrm{L})\ \frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta} \qquad (\neg\mathrm{R})\ \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}$$

$$(\wedge\mathrm{R})\ \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}$$

$$(\vee\mathrm{R})\ \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}$$

Example Derivation (ctd.)

$$(\vee\mathrm{R})\ \frac{\neg(P \wedge Q) \vdash \neg P, \neg Q}{\neg(P \wedge Q) \vdash \neg P \vee \neg Q}$$

$$(\neg\mathrm{L})\ \frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta} \qquad (\neg\mathrm{R})\ \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}$$

$$(\wedge\mathrm{R})\ \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}$$

$$(\vee\mathrm{R})\ \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}$$

Example Derivation (ctd.)

$$(\lnot L)\;\dfrac{\vdash P \land Q, \lnot P, \lnot Q}{(\lor R)\;\dfrac{\lnot(P \land Q) \vdash \lnot P, \lnot Q}{\lnot(P \land Q) \vdash \lnot P \lor \lnot Q}}$$

$$(\lnot L)\;\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \lnot\varphi \vdash \Delta} \qquad (\lnot R)\;\frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \lnot\varphi, \Delta}$$

$$(\land R)\;\frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \land \psi, \Delta}$$

$$(\lor R)\;\frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \lor \psi, \Delta}$$

Example Derivation (ctd.)

$$
\dfrac{\dfrac{\dfrac{P \vdash P \wedge Q, \neg Q}{\vdash P \wedge Q, \neg P, \neg Q}\,(\neg\mathrm{R})}{\neg(P \wedge Q) \vdash \neg P, \neg Q}\,(\neg\mathrm{L})}{\neg(P \wedge Q) \vdash \neg P \vee \neg Q}\,(\vee\mathrm{R})
$$

$$
(\neg\mathrm{L})\ \frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta} \qquad (\neg\mathrm{R})\ \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}
$$

$$
(\wedge\mathrm{R})\ \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}
$$

$$
(\vee\mathrm{R})\ \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}
$$

Example Derivation (ctd.)

$$
(\neg R)\ \dfrac{P, Q \vdash P \wedge Q}{P \vdash P \wedge Q, \neg Q}
$$

$$
(\neg R)\ \dfrac{P \vdash P \wedge Q, \neg Q}{\vdash P \wedge Q, \neg P, \neg Q}
$$

$$
(\neg L)\ \dfrac{\vdash P \wedge Q, \neg P, \neg Q}{\neg(P \wedge Q) \vdash \neg P, \neg Q}
$$

$$
(\vee R)\ \dfrac{\neg(P \wedge Q) \vdash \neg P, \neg Q}{\neg(P \wedge Q) \vdash \neg P \vee \neg Q}
$$

$$
(\neg L)\ \dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta} \qquad (\neg R)\ \dfrac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}
$$

$$
(\wedge R)\ \dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}
$$

$$
(\vee R)\ \dfrac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}
$$

Example Derivation (ctd.)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P, Q \vdash P \qquad P, Q \vdash Q}{P, Q \vdash P \wedge Q}\ (\wedge\mathrm{R})}{P \vdash P \wedge Q, \neg Q}\ (\neg\mathrm{R})}{\vdash P \wedge Q, \neg P, \neg Q}\ (\neg\mathrm{R})}{\neg(P \wedge Q) \vdash \neg P, \neg Q}\ (\neg\mathrm{L})}{\neg(P \wedge Q) \vdash \neg P \vee \neg Q}\ (\vee\mathrm{R})
$$

$$
(\neg\mathrm{L})\ \dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta} \qquad (\neg\mathrm{R})\ \dfrac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}
$$

$$
(\wedge\mathrm{R})\ \dfrac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}
$$

$$
(\vee\mathrm{R})\ \dfrac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}
$$

Inference Rules for the Propositional Connectives

Basic sequents: $\Gamma, \varphi \vdash \varphi, \Delta$
$\Gamma, \bot \vdash \Delta$

$$(\text{CUT})\ \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta}$$

$$(\neg\text{L})\ \frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta} \qquad (\neg\text{R})\ \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}$$

$$(\wedge\text{L})\ \frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta} \qquad (\wedge\text{R})\ \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}$$

$$(\vee\text{L})\ \frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta} \qquad (\vee\text{R})\ \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}$$

$$(\to\text{L})\ \frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \to \psi \vdash \Delta} \qquad (\to\text{R})\ \frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta}$$

Example Derivations

$$\vdash ((P \to Q) \to P) \to P$$

Example Derivations

$$(\to\mathrm{R})\ \frac{(P \to Q) \to P \vdash P}{\vdash ((P \to Q) \to P) \to P}$$

Example Derivations

$$
(\to\mathrm{L})\ \dfrac{\vdash P \to Q, P \qquad P \vdash P}{(P \to Q) \to P \vdash P}
$$

$$
(\to\mathrm{R})\ \dfrac{(P \to Q) \to P \vdash P}{\vdash ((P \to Q) \to P) \to P}
$$

Example Derivations

$$
\dfrac{\dfrac{\dfrac{P \vdash Q, P}{\vdash P \to Q, P}\,(\to\mathrm{R}) \qquad P \vdash P}{(P \to Q) \to P \vdash P}\,(\to\mathrm{L})}{\vdash ((P \to Q) \to P) \to P}\,(\to\mathrm{R})
$$

Example Derivations (ctd.)

$$\neg\neg P \vdash P$$

Example Derivations (ctd.)

$$(\neg\mathrm{L})\ \frac{\vdash \neg P, P}{\neg\neg P \vdash P}$$

Example Derivations (ctd.)

$$\dfrac{\dfrac{P \vdash P}{\vdash \neg P, P}\,(\neg\mathrm{R})}{\neg\neg P \vdash P}\,(\neg\mathrm{L})$$

Example Derivations (ctd.)

$$P \vee \bot \vdash P$$

Example Derivations (ctd.)

$$(\vee\mathrm{L})\ \frac{P \vdash P \qquad \bot \vdash P}{P \vee \bot \vdash P}$$

Inference Rules for the Quantifiers

$$(\forall\mathrm{L})\ \frac{\Gamma, \varphi[t/x] \vdash \Delta}{\Gamma, \forall x.\varphi \vdash \Delta} \qquad (\forall\mathrm{R})\ \frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \forall x.\varphi, \Delta}\ \text{if } x \notin \mathrm{FV}(\Gamma, \Delta)$$

$$(\exists\mathrm{L})\ \frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \exists x.\varphi \vdash \Delta}\ \text{if } x \notin \mathrm{FV}(\Gamma, \Delta) \qquad (\exists\mathrm{R})\ \frac{\Gamma \vdash \varphi[t/x], \Delta}{\Gamma \vdash \exists x.\varphi, \Delta}$$

Example Derivations

$$\forall x.P(x) \land Q(x) \vdash (\forall x.P(x)) \land (\forall x.Q(x))$$

Example Derivations

$$(\wedge\mathrm{R})\ \frac{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x) \qquad \forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}$$

Example Derivations

$$
(\wedge R)\ \dfrac{(\forall R)\ \dfrac{\forall x.P(x) \wedge Q(x) \vdash P(x)}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)} \qquad \forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}
$$

Example Derivations

$$
(\forall R)\ \dfrac{(\forall L)\ \dfrac{P(x) \wedge Q(x) \vdash P(x)}{\forall x.P(x) \wedge Q(x) \vdash P(x)}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)}
$$

$$
(\wedge R)\ \dfrac{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x) \qquad \forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}
$$

Example Derivations

$$
\begin{prooftree}
\AxiomC{$P(x), Q(x) \vdash P(x)$}
\LeftLabel{$(\wedge\mathrm{L})$}
\UnaryInfC{$P(x) \wedge Q(x) \vdash P(x)$}
\LeftLabel{$(\forall\mathrm{L})$}
\UnaryInfC{$\forall x. P(x) \wedge Q(x) \vdash P(x)$}
\LeftLabel{$(\forall\mathrm{R})$}
\UnaryInfC{$\forall x. P(x) \wedge Q(x) \vdash \forall x. P(x)$}
\AxiomC{$\forall x. P(x) \wedge Q(x) \vdash \forall x. Q(x)$}
\LeftLabel{$(\wedge\mathrm{R})$}
\BinaryInfC{$\forall x. P(x) \wedge Q(x) \vdash (\forall x. P(x)) \wedge (\forall x. Q(x))$}
\end{prooftree}
$$

Example Derivations

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{P(x), Q(x) \vdash P(x)}{P(x) \wedge Q(x) \vdash P(x)}\,(\wedge\mathrm{L})
    }{\forall x.P(x) \wedge Q(x) \vdash P(x)}\,(\forall\mathrm{L})
  }{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)}\,(\forall\mathrm{R})
  \qquad
  \dfrac{\forall x.P(x) \wedge Q(x) \vdash Q(x)}{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}\,(\forall\mathrm{R})
}{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}\,(\wedge\mathrm{R})
$$

Example Derivations

$$
(\wedge\mathrm{R})\ \dfrac{(\forall\mathrm{R})\ \dfrac{(\forall\mathrm{L})\ \dfrac{(\wedge\mathrm{L})\ \dfrac{P(x), Q(x) \vdash P(x)}{P(x) \wedge Q(x) \vdash P(x)}}{\forall x.P(x) \wedge Q(x) \vdash P(x)}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)} \qquad (\forall\mathrm{R})\ \dfrac{(\forall\mathrm{L})\ \dfrac{P(x) \wedge Q(x) \vdash Q(x)}{\forall x.P(x) \wedge Q(x) \vdash Q(x)}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}}{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}
$$

Example Derivations

$$
(\wedge\mathrm{R})\ \dfrac{
(\forall\mathrm{R})\ \dfrac{
(\forall\mathrm{L})\ \dfrac{
(\wedge\mathrm{L})\ \dfrac{P(x), Q(x) \vdash P(x)}{P(x) \wedge Q(x) \vdash P(x)}
}{\forall x.P(x) \wedge Q(x) \vdash P(x)}
}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)}
\qquad
(\forall\mathrm{R})\ \dfrac{
(\forall\mathrm{L})\ \dfrac{
(\wedge\mathrm{L})\ \dfrac{P(x), Q(x) \vdash Q(x)}{P(x) \wedge Q(x) \vdash Q(x)}
}{\forall x.P(x) \wedge Q(x) \vdash Q(x)}
}{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}
}{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}
$$

Example Derivations (ctd.)

The following derivation violates a side condition:

$$
\cfrac{\cfrac{\cfrac{P(x) \vdash P(x)}{\exists x. P(x) \vdash P(x)}\;(\exists\mathrm{L})}{\exists x. P(x) \vdash \forall x. P(x)}\;(\forall\mathrm{R})}{\vdash (\exists x. P(x)) \to (\forall x. P(x))}\;(\to\mathrm{R})
$$

Inference Rules for Equality

Basic sequent: $\Gamma \vdash t \doteq t, \Delta$ $\qquad$ $\text{(SUBST)}\ \dfrac{s \doteq t, \Gamma[s/x] \vdash \Delta[s/x]}{s \doteq t, \Gamma[t/x] \vdash \Delta[t/x]}$

$$s \doteq t \vdash t \doteq s$$

Inference Rules for Equality

Basic sequent: $\Gamma \vdash t \doteq t, \Delta$

$$\text{(SUBST)}\ \frac{s \doteq t, \Gamma[s/x] \vdash \Delta[s/x]}{s \doteq t, \Gamma[t/x] \vdash \Delta[t/x]}$$

$$s \doteq t \vdash \underbrace{t \doteq s}_{\equiv (x \doteq s)[t/x]}$$

Inference Rules for Equality

Basic sequent: $\Gamma \vdash t \doteq t, \Delta$

$$(\textsc{subst})\ \frac{s \doteq t, \Gamma[s/x] \vdash \Delta[s/x]}{s \doteq t, \Gamma[t/x] \vdash \Delta[t/x]}$$

$$(\textsc{subst})\ \frac{s \doteq t \vdash \overbrace{s \doteq s}^{\equiv (x \doteq s)[s/x]}}{s \doteq t \vdash \underbrace{t \doteq s}_{\equiv (x \doteq s)[t/x]}}$$

Soundness and Completeness

We write $\Gamma \vdash_{\mathrm{LK}} \Delta$ to mean that there is a derivation of $\Gamma \vdash \Delta$.

Theorem 15 (Soundness of LK)

The system LK is sound: If $\Gamma \vdash_{\mathrm{LK}} \varphi$ then $\Gamma \models \varphi$.

Theorem 16 (Consistency of LK)

The system LK is consistent: There is a formula φ such that $\nvdash_{\mathrm{LK}} \varphi$.

Theorem 17 (Completeness of LK)

The system LK is complete: If $\Gamma \models \varphi$ then $\Gamma \vdash_{\mathrm{LK}} \varphi$.