Logic
Part I: Propositional Logic

Max Schäfer

Formosan Summer School on Logic, Language, and Computation 2010

Organisation

Lecturer:	Max Schäfer (Schaefer) max.schaefer@comlab.ox.ac.uk
Lectures:	Monday 28th June, 2:00pm – 5:00pm Wednesday 30th June, 9:30am – 12:30pm Friday 2nd July, 9:30am – 12:30pm
Homework:	after every lecture
Exam	Friday 9th July, 9:30am – 12:30pm material of first two lectures *only*

What Is Logic?

- this course is about *formal logic*
- investigate principles of reasoning, independently of particular language, mindset, or philosophy
- different logical systems for different kinds of reasoning
- three basic components of a logical system:
    1. *formal language*
    2. *semantics*
    3. *deductive system*

Principles of Classical Logic

- classical logic aims to model reasoning about truth
- logical formulas represent statements that are either true or false
- proving a formula means showing that it is true
- sometimes this is easy

$$\sqrt{2} \notin \mathbb{Q}$$

- sometimes it is hard

$$\forall n. n > 2 \rightarrow \neg(\exists a, b, c. a^n + b^n = c^n)$$

- proving a formula does not "make" it true, it just *demonstrates* its truth

Propositional Logic

命題邏輯

Principles of Propositional Logic

- propositional logic talks about *propositions*
- a proposition is a sentence that is either true or false
- some propositions are *atomic*; represented by capital letters $P, Q, R, \ldots$
- other propositions are composed from simpler ones using *connectives* such as $\wedge, \vee, \ldots$

The Formal Language of Propositional Logic

- assume we have an alphabet $\mathcal{R}$ of *propositional letters*, assumed to contain at least the capital letters P, Q, R, ...
- the language of *formulas* of propositional logic is given by the following grammar:

$$\varphi ::= \mathcal{R} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi$$

Intuitive Meaning of Propositional Logic

formula	**reading**	**is true if...**
P	P	the proposition represented by P is true
$\bot$	假, falsity, bottom	never true
$P \wedge Q$	P 與 Q, conjunction	P is true, and Q is also true
$P \vee Q$	P 或 Q, disjunction	P is true, or Q is true, or both
$P \to Q$	P 就 Q, implication	it is not the case that P is true and Q is false

Example Formulas

- P, Q
- $\bot$
- $P \wedge \bot$
- $P \wedge Q \vee P$; interpreted as $(P \wedge Q) \vee P$

  not as $P \wedge (Q \vee P)$
- $P \rightarrow P \vee Q$; interpreted as $P \rightarrow (P \vee Q)$

  not as $(P \rightarrow P) \vee Q$

Operator Precedence

$\wedge$ binds tighter than $\vee$; $\vee$ tighter than $\rightarrow$

Example Formulas (ctd.)

- $P \land Q \land R$; interpreted as $(P \land Q) \land R$
  not as $P \land (Q \land R)$
- $P \lor Q \lor R$; interpreted as $(P \lor Q) \lor R$
  not as $P \lor (Q \lor R)$
- $P \to P \to Q$; interpreted as $P \to (P \to Q)$
  not as $(P \to P) \to Q$

Operator Associativity

$\land$ and $\lor$ associate to the left, $\to$ to the right

Defined Connectives and Syntactic Equality

- other connectives can be defined in terms of the basic ones:
  - $\neg\varphi := \varphi \to \bot$

    $\neg P$: 非 P, negation
  - $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$

    $P \leftrightarrow Q$: P 若且唯若 Q, equivalence, bi-implication
  - $\top := \bot \to \bot$

    $\top$: 眞, truth, top
- $\neg$ and $\leftrightarrow$ are not real connectives, but only abbreviations; e.g., $\neg P \equiv P \to \bot$ (the *same* formula)

Precedence and Associativity

$\neg$ binds tighter than $\wedge$; $\leftrightarrow$ less tight than $\vee$, associates to the left

Truth Value Semantics

- in general, to know whether a formula φ is true, we need to know whether its propositional letters are true
- need a truth value assignment (interpretation) $I \colon \mathcal{R} \to \mathcal{B}$, where $\mathcal{B} := \{\mathtt{T}, \mathtt{F}\}$
- given an interpretation I, define the semantics $[\![\varphi]\!]_I$ of a formula φ:
  1. for $r \in \mathcal{R}$: $[\![r]\!]_I := I(r)$.
  2. $[\![\bot]\!]_I := \mathtt{F}$.
  3. $[\![\varphi \wedge \psi]\!]_I := \mathtt{T}$ if $[\![\varphi]\!]_I = \mathtt{T}$ and $[\![\psi]\!]_I = \mathtt{T}$, else $[\![\varphi \wedge \psi]\!]_I := \mathtt{F}$.
  4. $[\![\varphi \vee \psi]\!]_I := \mathtt{F}$ if $[\![\varphi]\!]_I = \mathtt{F}$ and $[\![\psi]\!]_I = \mathtt{F}$, else $[\![\varphi \vee \psi]\!]_I := \mathtt{T}$.
  5. $[\![\varphi \to \psi]\!]_I := \mathtt{F}$ if $[\![\varphi]\!]_I = \mathtt{T}$ and $[\![\psi]\!]_I = \mathtt{F}$, else $[\![\varphi \to \psi]\!]_I := \mathtt{T}$.

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    5. $[\![\varphi \to \psi]\!]_I := \mathtt{F}$ if $[\![\varphi]\!]_I = \mathtt{T}$ and $[\![\psi]\!]_I = \mathtt{F}$, else $[\![\varphi \to \psi]\!]_I := \mathtt{T}$.

Derived Truth Values

Lemma

For any interpretation I and formulas φ and ψ we have

- $[\![\top]\!]_I = \mathtt{T}$.
- $[\![\neg\varphi]\!]_I = \mathtt{T}$ if $[\![\varphi]\!]_I = \mathtt{F}$, else $[\![\neg\varphi]\!]_I = \mathtt{F}$.
- $[\![\varphi \leftrightarrow \psi]\!]_I = \mathtt{T}$ if $[\![\varphi]\!]_I = [\![\psi]\!]_I$, else $[\![\varphi \leftrightarrow \psi]\!]_I = \mathtt{F}$.

Validity and Satisfiability

- $I \models \varphi$ ("I is a model for φ"): $[\![\varphi]\!]_I = \mathtt{T}$
- φ is satisfiable: there is I with $I \models \varphi$
- φ is valid: for all I we have $I \models \varphi$
- $\varphi \Rightarrow \psi$ ("φ entails ψ"): whenever $I \models \varphi$ also $I \models \psi$
- $\varphi \Leftrightarrow \psi$ ("φ and ψ are equivalent"): both $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$

Examples

- $P \vee \neg P$ is valid
- $P \to \neg P$ is satisfiable
- $P \to Q \Leftrightarrow \neg P \vee Q$

Properties of Validity and Satisfiability

Theorem

φ is valid iff $\neg\varphi$ is unsatisfiable iff $\varphi \Leftrightarrow \top$;
furthermore, $\varphi \Leftrightarrow \psi$ iff $[\![\varphi]\!]_I = [\![\psi]\!]_I$ for every interpretation I.

Example: $\neg\neg\varphi \Leftrightarrow \varphi$

Notational Caveat

$\varphi \equiv \psi$ and $\varphi \Leftrightarrow \psi$ do *not* mean the same!

- $\equiv$ is syntactic equality; different ways of writing, same formula

  $(P \wedge Q) \wedge \neg R \equiv P \wedge Q \wedge \neg R \equiv P \wedge Q \wedge (R \rightarrow \bot)$

- $\Leftrightarrow$ is semantic equality; different formulas, same semantics

  $\neg\neg P \leftrightarrow P$, but not $\neg\neg P \equiv P$

Propositional Letters in a Formula

- define set $\mathrm{PL}(\varphi)$ of propositional letters that occur in a formula φ:
  - $\mathrm{PL}(r) = \{r\}$, for every $r \in \mathcal{R}$
  - $\mathrm{PL}(\bot) = \emptyset$
  - $\mathrm{PL}(\varphi \wedge \psi) = \mathrm{PL}(\varphi \vee \psi) = \mathrm{PL}(\varphi \to \psi) = \mathrm{PL}(\varphi) \cup \mathrm{PL}(\psi)$

Example

$$\mathrm{PL}(((P \to Q) \to P) \to P) \quad =$$
$$\mathrm{PL}(\top) \quad =$$

Propositional Letters in a Formula

- define set $\mathrm{PL}(\varphi)$ of propositional letters that occur in a formula φ:
    - $\mathrm{PL}(r) = \{r\}$, for every $r \in \mathcal{R}$
    - $\mathrm{PL}(\bot) = \emptyset$
    - $\mathrm{PL}(\varphi \land \psi) = \mathrm{PL}(\varphi \lor \psi) = \mathrm{PL}(\varphi \to \psi) = \mathrm{PL}(\varphi) \cup \mathrm{PL}(\psi)$

Example

$$
\begin{aligned}
\mathrm{PL}(((P \to Q) \to P) \to P) &= \{P, Q\} \\
\mathrm{PL}(\top) &=
\end{aligned}
$$

Propositional Letters in a Formula

- define set $\mathrm{PL}(\varphi)$ of propositional letters that occur in a formula φ:
  - $\mathrm{PL}(r) = \{r\}$, for every $r \in \mathcal{R}$
  - $\mathrm{PL}(\bot) = \emptyset$
  - $\mathrm{PL}(\varphi \wedge \psi) = \mathrm{PL}(\varphi \vee \psi) = \mathrm{PL}(\varphi \to \psi) = \mathrm{PL}(\varphi) \cup \mathrm{PL}(\psi)$

Example

$$\begin{array}{lcl} \mathrm{PL}(((P \to Q) \to P) \to P) & = & \{P, Q\} \\ \mathrm{PL}(\top) & = & \emptyset \end{array}$$

Agreement Lemma

Lemma

For every formula φ and interpretations I_1, I_2 such that $I_1(P) = I_2(P)$ for every $P \in \mathrm{PL}(\varphi)$, we have

$$[\![\varphi]\!]_{I_1} = [\![\varphi]\!]_{I_2}$$

Propositional letters that don't occur in a formula do not matter when determining its semantics.

Truth Tabling

- for every formula φ, $\mathrm{PL}(\varphi)$ is finite, say $|\mathrm{PL}(\varphi)| = n$
- every one of these n variables could be either true or false; this gives 2^n combinations
- to know whether φ is valid, we only need to try them all out!

Example

We can use truth tables to show:

- $\models P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$

P	Q	$P \wedge Q$	$P \vee Q$	$P \leftrightarrow Q$	$P \wedge Q \leftrightarrow P \vee Q$	$P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$
F	F					
F	T					
T	F					
T	T					

- $P \vee Q \Leftrightarrow \neg P \rightarrow Q$

Example

We can use truth tables to show:

- $\models P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$

P	Q	$P \wedge Q$	$P \vee Q$	$P \leftrightarrow Q$	$P \wedge Q \leftrightarrow P \vee Q$	$P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$
F	F	F				
F	T	F				
T	F	F				
T	T	T				

- $P \vee Q \Leftrightarrow \neg P \rightarrow Q$

Example

We can use truth tables to show:

- $\models P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$

P	Q	$P \wedge Q$	$P \vee Q$	$P \leftrightarrow Q$	$P \wedge Q \leftrightarrow P \vee Q$	$P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$
F	F	F	F			
F	T	F	T			
T	F	F	T			
T	T	T	T			

- $P \vee Q \Leftrightarrow \neg P \rightarrow Q$

Example

We can use truth tables to show:

- $\models P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$

P	Q	$P \wedge Q$	$P \vee Q$	$P \leftrightarrow Q$	$P \wedge Q \leftrightarrow P \vee Q$	$P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$
F	F	F	F	T		
F	T	F	T	F		
T	F	F	T	F		
T	T	T	T	T		

- $P \vee Q \Leftrightarrow \neg P \rightarrow Q$

Example

We can use truth tables to show:

- $\models P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$

P	Q	$P \wedge Q$	$P \vee Q$	$P \leftrightarrow Q$	$P \wedge Q \leftrightarrow P \vee Q$	$P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$
F	F	F	F	T	T	
F	T	F	T	F	F	
T	F	F	T	F	F	
T	T	T	T	T	T	

- $P \vee Q \Leftrightarrow \neg P \rightarrow Q$

Example

We can use truth tables to show:

- $\models P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$

P	Q	$P \wedge Q$	$P \vee Q$	$P \leftrightarrow Q$	$P \wedge Q \leftrightarrow P \vee Q$	$P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$
F	F	F	F	T	T	T
F	T	F	T	F	F	T
T	F	F	T	F	F	T
T	T	T	T	T	T	T

- $P \vee Q \Leftrightarrow \neg P \rightarrow Q$

Some Properties of Equivalence

Equivalence is

- reflexive: $\varphi \Leftrightarrow \varphi$
- symmetric: if $\varphi \Leftrightarrow \psi$ then $\psi \Leftrightarrow \varphi$
- transitive: if $\varphi \Leftrightarrow \psi$ and $\psi \Leftrightarrow \chi$ then $\varphi \Leftrightarrow \chi$

Connection between $\leftrightarrow$ and $\Leftrightarrow$

$\models \varphi \leftrightarrow \psi$ iff $\varphi \Leftrightarrow \psi$

Substitution for Propositional Letters

We substitute a formula ϑ for all occurrences of $r \in \mathcal{R}$ in a formula φ, written as $\varphi[\vartheta/r]$ as follows:

- if φ is some $r' \in \mathcal{R}$, then
  - $\varphi[\vartheta/r] := \vartheta$ if $r = r'$
  - $\varphi[\vartheta/r] := r'$ otherwise
- $\bot[\vartheta/r] := \bot$
- $(\psi \wedge \chi)[\vartheta/r] := \psi[\vartheta/r] \wedge \chi[\vartheta/r]$
- $(\psi \vee \chi)[\vartheta/r] := \psi[\vartheta/r] \vee \chi[\vartheta/r]$
- $(\psi \to \chi)[\vartheta/r] := \psi[\vartheta/r] \to \chi[\vartheta/r]$.

Tautologies for free!

Substitution in Tautologies

If $\models \varphi$ then $\models \varphi[\psi/r]$.

- Once we have shown that $\models P \vee \neg P$, we know that $\models \varphi \vee \neg\varphi$ for any φ.
- If $\varphi_1 \Leftrightarrow \varphi_2$, then also $\varphi_1[\psi/r] \Leftrightarrow \varphi_2[\psi/r]$.

Properties of Substitution (II)

Leibniz' Law

If $\psi_1 \Leftrightarrow \psi_2$ then $\varphi[\psi_1/r] \Leftrightarrow \varphi[\psi_2/r]$.

- $\varphi \wedge (\psi \vee \chi) \Leftrightarrow \varphi \wedge (\neg\psi \rightarrow \chi)$
- We can eliminate $\vee$ from any formula!

Truth Functions

- *truth function* of arity n: function from $\mathcal{B}^n$ to $\mathcal{B}$
- formulas give rise to truth functions:
  - for $r_1, \ldots, r_n \in \mathcal{R}$ and $x_1, \ldots, x_n \in \mathcal{B}$, define interpretation

$$I_{r_1:=x_1,\ldots,r_n:=x_n}(r) := \begin{cases} x_i & r = r_i \\ \mathtt{F} & \text{else} \end{cases}$$

  - for formula φ with $\mathrm{PL}(\varphi) = \{r_1, \ldots, r_n\}$ define truth function $f_\varphi \colon \mathcal{B}^n \to \mathcal{B}$ by

$$f_\varphi(x_1, \ldots, x_n) := [\![\varphi]\!]_{I_{r_1:=x_1,\ldots,r_n:=x_n}}$$

Example: $f_{P\wedge Q}(x_1, x_2) = \begin{cases} \mathtt{T} & \text{if } x_1 = x_2 = \mathtt{T} \\ \mathtt{F} & \text{else} \end{cases}$

Functional Completeness

Functional Completeness

A set O of operators is *functionally complete* if, for every $f : \mathcal{B}^n \to \mathcal{B}$, there is a formula φ_f using only operators from O such that $f_{\varphi_f} = f$.

- $\{\bot, \to, \vee, \wedge\}$ is functionally complete
- so are $\{\bot, \to, \wedge\}$ and $\{\bot, \to\}$
- but $\{\bot, \vee\}$ is not

Functional Completeness (ctd.)

P	Q	R	f
F	F	F	F
F	F	T	F
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

$$\begin{aligned} \varphi_f \quad := \quad & \neg P \wedge Q \wedge R \\ \vee \quad & P \wedge \neg Q \wedge R \\ \vee \quad & P \wedge Q \wedge \neg R \\ \vee \quad & P \wedge Q \wedge R \end{aligned}$$

Functional Completeness (ctd.)

P	Q	R	f
F	F	F	F
F	F	T	F
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

$$
\begin{aligned}
\varphi_f \quad := \quad & \neg P \wedge Q \wedge R \\
\vee \quad & P \wedge \neg Q \wedge R \\
\vee \quad & P \wedge Q \wedge \neg R \\
\vee \quad & P \wedge Q \wedge R
\end{aligned}
$$

Functional Completeness (ctd.)

P	Q	R	f
F	F	F	F
F	F	T	F
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

$$
\begin{aligned}
\varphi_f \quad := \quad & \neg P \wedge Q \wedge R \\
\vee \quad & P \wedge \neg Q \wedge R \\
\vee \quad & P \wedge Q \wedge \neg R \\
\vee \quad & P \wedge Q \wedge R
\end{aligned}
$$

Functional Completeness (ctd.)

P	Q	R	f
F	F	F	F
F	F	T	F
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

$$
\begin{aligned}
\varphi_f \quad := \quad & \neg P \wedge Q \wedge R \\
\vee \quad & P \wedge \neg Q \wedge R \\
\vee \quad & P \wedge Q \wedge \neg R \\
\vee \quad & P \wedge Q \wedge R
\end{aligned}
$$

Functional Completeness (ctd.)

P	Q	R	f
F	F	F	F
F	F	T	F
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

$$
\begin{array}{rcl}
\varphi_f & := & \neg P \wedge Q \wedge R \\
& \vee & P \wedge \neg Q \wedge R \\
& \vee & P \wedge Q \wedge \neg R \\
& \vee & P \wedge Q \wedge R
\end{array}
$$

Calculational Logic

演算邏輯

Principles of Calculational Logic

- calculational logic is a deductive system for propositional logic
- *not* a "new" logic
- idea: calculate with formulas to establish their truth
- avoid case distinctions, truth tables
- make use of a set of *laws*: tautologies of the form

$$\varphi \leftrightarrow \psi$$

- replace equivalent formulas

Example Derivation

Laws:

- Associativity of $\leftrightarrow$: $((P \leftrightarrow Q) \leftrightarrow R) \leftrightarrow (P \leftrightarrow (Q \leftrightarrow R))$
- Symmetry of $\leftrightarrow$: $P \leftrightarrow Q \leftrightarrow Q \leftrightarrow P$
- Unfolding $\neg$: $\neg P \leftrightarrow P \leftrightarrow \bot$

Derivation:

$$\neg(P \leftrightarrow Q)$$

Example Derivation

Laws:

- Associativity of $\leftrightarrow$: $((P \leftrightarrow Q) \leftrightarrow R) \leftrightarrow (P \leftrightarrow (Q \leftrightarrow R))$
- Symmetry of $\leftrightarrow$: $P \leftrightarrow Q \leftrightarrow Q \leftrightarrow P$
- Unfolding $\neg$: $\neg(P \leftrightarrow Q) \leftrightarrow ((P \leftrightarrow Q) \leftrightarrow \bot)$

Derivation:

$$\neg(P \leftrightarrow Q)$$

Example Derivation

Laws:

- Associativity of $\leftrightarrow$: $((P \leftrightarrow Q) \leftrightarrow R) \leftrightarrow (P \leftrightarrow (Q \leftrightarrow R))$
- Symmetry of $\leftrightarrow$: $P \leftrightarrow Q \leftrightarrow Q \leftrightarrow P$
- Unfolding $\neg$: $\neg(P \leftrightarrow Q) \leftrightarrow ((P \leftrightarrow Q) \leftrightarrow \bot)$

Derivation:

$$
\begin{array}{ll}
 & \neg(P \leftrightarrow Q) \\
\Leftrightarrow & \qquad \{ \text{ Unfolding } \neg \text{ } \} \\
 & (P \leftrightarrow Q) \leftrightarrow \bot
\end{array}
$$

Example Derivation

Laws:

- Associativity of $\leftrightarrow$: $((P \leftrightarrow Q) \leftrightarrow \bot) \leftrightarrow (P \leftrightarrow (Q \leftrightarrow \bot))$
- Symmetry of $\leftrightarrow$: $P \leftrightarrow Q \leftrightarrow Q \leftrightarrow P$
- Unfolding $\neg$: $\neg P \leftrightarrow P \leftrightarrow \bot$

Derivation:

$$
\begin{array}{ll}
 & \neg(P \leftrightarrow Q) \\
\Leftrightarrow & \qquad \{ \text{ Unfolding } \neg \text{ } \} \\
 & (P \leftrightarrow Q) \leftrightarrow \bot
\end{array}
$$

Example Derivation

Laws:

- Associativity of $\leftrightarrow$: $((P \leftrightarrow Q) \leftrightarrow \bot) \leftrightarrow (P \leftrightarrow (Q \leftrightarrow \bot))$
- Symmetry of $\leftrightarrow$: $P \leftrightarrow Q \leftrightarrow Q \leftrightarrow P$
- Unfolding $\neg$: $\neg P \leftrightarrow P \leftrightarrow \bot$

Derivation:

$$
\begin{array}{ll}
 & \neg(P \leftrightarrow Q) \\
\Leftrightarrow & \qquad \{ \text{ Unfolding } \neg \ \} \\
 & (P \leftrightarrow Q) \leftrightarrow \bot \\
\Leftrightarrow & \qquad \{ \text{ Associativity of } \leftrightarrow \ \} \\
 & P \leftrightarrow (Q \leftrightarrow \bot)
\end{array}
$$

Example Derivation

Laws:

- Associativity of $\leftrightarrow$: $((P \leftrightarrow Q) \leftrightarrow R) \leftrightarrow (P \leftrightarrow (Q \leftrightarrow R))$
- Symmetry of $\leftrightarrow$: $P \leftrightarrow Q \leftrightarrow Q \leftrightarrow P$
- Unfolding $\neg$: $\neg Q \leftrightarrow Q \leftrightarrow \bot$

Derivation:

$$
\begin{array}{ll}
 & \neg(P \leftrightarrow Q) \\
\Leftrightarrow & \qquad \{ \text{ Unfolding } \neg \text{ } \} \\
 & (P \leftrightarrow Q) \leftrightarrow \bot \\
\Leftrightarrow & \qquad \{ \text{ Associativity of } \leftrightarrow \text{ } \} \\
 & P \leftrightarrow (Q \leftrightarrow \bot)
\end{array}
$$

Example Derivation

Laws:

- Associativity of $\leftrightarrow$: $((P \leftrightarrow Q) \leftrightarrow R) \leftrightarrow (P \leftrightarrow (Q \leftrightarrow R))$
- Symmetry of $\leftrightarrow$: $P \leftrightarrow Q \leftrightarrow Q \leftrightarrow P$
- Unfolding $\neg$: $\neg Q \leftrightarrow Q \leftrightarrow \bot$

Derivation:

$$
\begin{array}{ll}
 & \neg(P \leftrightarrow Q) \\
\Leftrightarrow & \qquad \{ \text{ Unfolding } \neg \text{ } \} \\
 & (P \leftrightarrow Q) \leftrightarrow \bot \\
\Leftrightarrow & \qquad \{ \text{ Associativity of } \leftrightarrow \text{ } \} \\
 & P \leftrightarrow (Q \leftrightarrow \bot) \\
\Leftrightarrow & \qquad \{ \text{ Folding } \neg \text{ } \} \\
 & P \leftrightarrow \neg Q
\end{array}
$$

Example Derivation

Laws:

- Associativity of $\leftrightarrow$: $((P \leftrightarrow Q) \leftrightarrow R) \leftrightarrow (P \leftrightarrow (Q \leftrightarrow R))$
- Symmetry of $\leftrightarrow$: $P \leftrightarrow Q \leftrightarrow Q \leftrightarrow P$
- Unfolding $\neg$: $\neg P \leftrightarrow P \leftrightarrow \bot$

Derivation:

$$\begin{array}{cl} & \underline{\neg(P \leftrightarrow Q)} \\ \Leftrightarrow & \qquad \{ \text{ Unfolding } \neg \text{ } \} \\ & P \leftrightarrow \underline{Q \leftrightarrow \bot} \\ \Leftrightarrow & \qquad \{ \text{ Folding } \neg \text{ } \} \\ & P \leftrightarrow \neg Q \end{array}$$

By transitivity: $\neg(P \leftrightarrow Q) \Leftrightarrow P \leftrightarrow \neg Q$

Laws for $\top$ and $\vee$

- Unfolding $\top$: $\top \leftrightarrow P \leftrightarrow P$
- Idempotence of $\vee$: $P \vee P \leftrightarrow P$
- Symmetry of $\vee$: $P \vee Q \leftrightarrow Q \vee P$
- Associativity of $\vee$: $P \vee (Q \vee R) \leftrightarrow (P \vee Q) \vee R$
- Distributivity of $\vee$: $P \vee (Q \leftrightarrow R) \leftrightarrow P \vee Q \leftrightarrow P \vee R$
- Excluded Middle: $P \vee \neg P \leftrightarrow \top$

Examples: $P \vee \bot \Leftrightarrow P$, $\models P \vee \top$

Law for $\wedge$

Golden Rule: $P \wedge Q \leftrightarrow P \leftrightarrow Q \leftrightarrow P \vee Q$

P	Q	$P \wedge Q$	$P \vee Q$	$P \leftrightarrow Q$	$P \wedge Q \leftrightarrow P \vee Q$	$P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$
F	F	F	F	T	T	T
F	T	F	T	F	F	T
T	F	F	T	F	F	T
T	T	T	T	T	T	T

"$P \wedge Q$ is true iff $P \leftrightarrow Q$ and $P \vee Q$ have the same truth value."

Example: $P \wedge (P \vee Q) \Leftrightarrow P$

Law for $\wedge$

Golden Rule: $P \wedge Q \leftrightarrow P \leftrightarrow Q \leftrightarrow P \vee Q$

P	Q	$P \wedge Q$	$P \vee Q$	$P \leftrightarrow Q$	$P \wedge Q \leftrightarrow P \vee Q$	$P \wedge Q \leftrightarrow P \vee Q \leftrightarrow (P \leftrightarrow Q)$
F	F	F	F	T	T	T
F	T	F	T	F	F	T
T	F	F	T	F	F	T
T	T	T	T	T	T	T

"$P \wedge Q$ is true iff $P \leftrightarrow Q$ and $P \vee Q$ have the same truth value."

Example: $P \wedge (P \vee Q) \Leftrightarrow P$

Law for $\rightarrow$ and Substitution Law

- Unfolding $\rightarrow$: $P \rightarrow Q \leftrightarrow Q \leftrightarrow P \vee Q$
- Substitution: $(P \leftrightarrow Q) \wedge \varphi[P/R] \leftrightarrow (P \leftrightarrow Q) \wedge \varphi[Q/R]$

Example: $\top \rightarrow P \leftrightarrow P$, $P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge Q$

The Island of Knights and Knaves (騎士與惡棍之島)

- on an island, there are two kinds of inhabitants: knights and knaves
- knives always speak the truth, knaves always lie
- assume inhabitant A says: "If you ask B, he will say he is a knight."
- what can we infer about A? what about B?

Knights and Knaves in Propositional Logic

- propositional letters represent identity of inhabitants
- let A mean "A is a knight"; then $\neg A$ is "A is a knave"
- statements about who is a knight or knave become propositional formulas
- assume A says φ:
  - if A is a knight, then φ is true
  - if A is a knave, then φ is false

  So whenever A says φ, we have $A \leftrightarrow \varphi$!
- "A says: B says he is a knight." is $A \leftrightarrow B \leftrightarrow B$

Statements about Knights and Knaves

- A says he is a knight.
- A says he is a knave.
- A and B are of the same kind.
- A says: "I am of the same kind as B".

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- A says: "I am of the same kind as B". $A \leftrightarrow (A \leftrightarrow B)$

Finding a Treasure on the Island

Legend has it there is a treasure on the island. We want to find out whether that is true.

Let the propositional letter T stand for "there is a treasure". Assume we meet inhabitant A. What question Q should we ask him to find out whether T is true?

- A answers Q with "yes": $A \leftrightarrow Q$
- A answers Q with "yes" iff there is a treasure: $A \leftrightarrow Q \leftrightarrow T$
- this is the same as $Q \leftrightarrow (A \leftrightarrow T)$
- So we should ask "Does 'there is a treasure on this island' equivale that you are a knight?"

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Some More Logic Puzzles

Assume A says any of the following things; what can you deduce about A and B?

- If I am a knight, then so is B.
- If B is a knight, then so am I.
- If I am a knave, then B is a knight.
- If I am a knight, then B is a knave.
- If B is a knave, then I am a knave.
- B says one of us is a knight.