Flolac 2010
Operational Semantics
Solution for Assignment 2: SOS , Due date: July 2

1. Specify the semantics of the construct "`repeat S until b`" in the style of SOS. The semantics of the repeat-construct is not allowed to rely on the existence of a while-construct in the language.

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<repeat S until b, s> →
   < S;if b then skip else repeat S until b end, s>
```

2. (Exercise 2.21) Non-interference of statements:
Prove that

if $\langle S_1, s\rangle \Rightarrow^k s'$ then $\langle S_1;S_2, s\rangle \Rightarrow^k (S_2, s')$

that is, the execution of S1 is not influenced by the statement following it.
(Hint: by induction on the length of derivation sequence)

We do the proof by induction on the length of the derivation sequence.

Base case: $k = 0$, the property holds as $\langle S_1, \sigma\rangle \to_1^0 \sigma'$ is not a valid transition.

Induction step: we assume that the property holds for $k \leq m$ and prove it for $m + 1$. Thus, we assume $\langle S_1, \sigma\rangle \to_1^{m+1} \sigma'$, which can be written as $\langle S_1, \sigma\rangle \to_1 \gamma \to_1^m \sigma'$ for some intermediate configuration γ. Now we have to make a case distinction depending on whether S_1 was executed in one or in multiple steps.

1. γ was obtained by executing S_1 in one step by transition $\langle S_1, \sigma\rangle \to_1 \sigma'$. Using this transition we can construct a derivation tree for transition $\langle S_1; S_2, \sigma\rangle \to_1 \langle S_2, \sigma'\rangle$. (In this case $m = 0$ and $\gamma = \sigma'$.)
2. γ was obtained by completing the first step of the execution of S_1. In this case we get derivation sequence $\langle S_1, \sigma\rangle \to_1 \langle S_1', \sigma''\rangle \to_1^m \sigma'$ for some statement S_1' and state σ''. Using the induction hypothesis on $\langle S_1', \sigma''\rangle \to_1^m \sigma'$ we get $\langle S_1'; S_2, \sigma''\rangle \to_1^m \langle S_2, \sigma'\rangle$. Using these results we can construct derivation sequence

$$\langle S_1; S_2, \sigma\rangle \to_1 \langle S_1'; S_2, \sigma''\rangle \to_1^m \langle S_2, \sigma'\rangle.$$

The validity of the first step is given by the following derivation tree:

$$\frac{\langle S_1, \sigma\rangle \to_1 \langle S_1', \sigma''\rangle}{\langle S_1; S_2, \sigma\rangle \to_1 \langle S_1'; S_2, \sigma''\rangle}$$

3. Prove that "S_1; (S_2; S_3)" and "(S_1; S_2); S_3" are semantically equivalent according to the SOS of While. Note that one direction (➔) of proof is good enough.
(Hint: by Lemma 2.19 and Exercise 2.21)

Direction $\Longrightarrow$

Case a: all three sub-statements terminate.
Using Lemma 2.19 we can get derivation sequences $\langle \mathtt{S}_1, \sigma\rangle \rightarrow_1^* \sigma''$ and $\langle \mathtt{S}_2;\mathtt{S}_3, \sigma''\rangle \rightarrow_1^* \sigma'$ for some state σ''. We can apply the lemma again on the second sequence and get $\langle \mathtt{S}_2, \sigma''\rangle \rightarrow_1^* \sigma'''$ and $\langle \mathtt{S}_3, \sigma'''\rangle \rightarrow_1^* \sigma'$ for some state σ'''. Using the result of Exercise 16 on sequence $\langle \mathtt{S}_1, \sigma\rangle \rightarrow_1^* \sigma''$ we get $\langle \mathtt{S}_1;\mathtt{S}_2, \sigma\rangle \rightarrow_1^* \langle \mathtt{S}_2, \sigma''\rangle$. This, combined with sequence $\langle \mathtt{S}_2, \sigma''\rangle \rightarrow_1^* \sigma'''$ yields $\langle \mathtt{S}_1;\mathtt{S}_2, \sigma\rangle \rightarrow_1^* \sigma'''$. Using the result of Exercise 16 on this sequence gives $\langle (\mathtt{S}_1;\mathtt{S}_2);\mathtt{S}_3, \sigma\rangle \rightarrow_1^* \langle \mathtt{S}_3, \sigma'''\rangle$. This, combined with sequence $\langle \mathtt{S}_3, \sigma'''\rangle \rightarrow_1^* \sigma'$ yields the required sequence $\langle (\mathtt{S}_1;\mathtt{S}_2);\mathtt{S}_3, \sigma\rangle \rightarrow_1^* \sigma'$.

Case b: any of the three sub-statements loops.
Since we have a looping statement in sequential composition with other statements, the whole composition will loop. Thus, both $\mathtt{S}_1;(\mathtt{S}_2;\mathtt{S}_3)$ and $(\mathtt{S}_1;\mathtt{S}_2);\mathtt{S}_3$ will loop.

Direction $\Longleftarrow$ Analogous.

4. (Extra credit) (Exercise 2.20) Suppose that (S_1; S_2, s) =>* (S_2, s'). Show (an example) that it is not necessarily the case that (S_1, s) =>* s'.

A counter-example is the following statement `skip; while true do x:=x+1 end` because we can construct the derivation sequence

$\langle \mathtt{skip;\ while\ true\ do\ x:=x+1\ end}, \sigma\rangle \rightarrow_1$
$\langle \mathtt{while\ true\ do\ x:=x+1\ end}, \sigma\rangle \rightarrow_1^3$
$\langle \mathtt{while\ true\ do\ x:=x+1\ end}, \sigma[\mathtt{x} \mapsto \sigma(\mathtt{x}) + 1]\rangle$

Thus,

$\langle \mathtt{skip;\ while\ true\ do\ x:=x+1\ end}, \sigma\rangle \rightarrow_1^* \langle \mathtt{while\ true\ do\ x:=x+1\ end}, \sigma[\mathtt{x} \mapsto \sigma(\mathtt{x})+1]\rangle$

holds, but $\langle \mathtt{skip}, \sigma\rangle \rightarrow_1^* \sigma[\mathtt{x} \mapsto \sigma(\mathtt{x}) + 1]$ is not a valid sequence.