Flolac 2010
Operational Semantics
Solution for Assignment 1, Due date: July 1

1. Prove that "`S1; (S2; S3)`" and "`(S1; S2); S3`" are semantically equivalent. Note that one direction of proof is good enough.

We have to show that for all σ, σ'

$$\langle S_1; (S_2; S_3), \sigma\rangle \to \sigma' \;\Leftrightarrow\; \langle (S_1; S_2); S_3), \sigma\rangle \to \sigma'$$

holds.

1. Direction $\Longrightarrow$: we know that there is a derivation tree for $\langle S_1; (S_2; S_3), \sigma\rangle \to \sigma'$ and have to show that there exists one for $\langle (S_1; S_2); S_3), \sigma\rangle \to \sigma'$.

The only derivation tree for $S_1; (S_2; S_3)$ is

$$\frac{\langle S_1, \sigma\rangle \to \sigma'', \quad \dfrac{\langle S_2, \sigma''\rangle \to \sigma''', \quad \langle S_3, \sigma'''\rangle \to \sigma'}{\langle S_2; S_3, \sigma''\rangle \to \sigma'}}{\langle S_1; (S_2; S_3), \sigma\rangle \to \sigma'}$$

Thus, we know that transitions $\langle S_1, \sigma\rangle \to \sigma''$, $\langle S_2, \sigma''\rangle \to \sigma'''$ and $\langle S_3, \sigma'''\rangle \to \sigma'$ hold. Putting them together in a different way, we can get the following derivation tree:

$$\frac{\dfrac{\langle S_1, \sigma\rangle \to \sigma'', \quad \langle S_2, \sigma''\rangle \to \sigma'''}{\langle S_1; S_2, \sigma\rangle \to \sigma'''}, \quad \langle S_3, \sigma'''\rangle \to \sigma'}{\langle (S_1; S_2); S_3, \sigma\rangle \to \sigma'}$$

2. Direction $\Longleftarrow$: Analogous.

2. Specify the semantics of the construct "`repeat S until b`" in the style of natural semantics. The semantics of the repeat-construct is not allowed to rely on the existence of a while-construct in the language.

For the `repeat` construct we need two rules

$$\frac{\langle s, \sigma\rangle \to \sigma'}{\langle \texttt{repeat}\ s\ \texttt{until}\ b, \sigma\rangle \to \sigma'}\ \mathcal{B}[\![b]\!]\sigma' = tt$$

$$\frac{\langle s, \sigma\rangle \to \sigma', \quad \langle \texttt{repeat}\ s\ \texttt{until}\ b, \sigma'\rangle \to \sigma''}{\langle \texttt{repeat}\ s\ \texttt{until}\ b, \sigma\rangle \to \sigma''}\ \mathcal{B}[\![b]\!]\sigma' = ff$$

3. (Bonus) Prove that "`repeat S until b`" and "`S;if b then skip else repeat S until b end`" are semantically equivalent.

The equivalence proof is as follows:

1. Direction $\Longrightarrow$: we assume there is a derivation tree T for

$$\langle \texttt{repeat}\ s\ \texttt{until}\ b, \sigma\rangle \rightarrow \sigma'$$

and have to show that there exists one for

$$\langle s\texttt{;if}\ b\ \texttt{then skip else repeat}\ s\ \texttt{until}\ b\ \texttt{end}, \sigma\rangle \rightarrow \sigma'.$$

We make a case split on the value of $\mathcal{B}[\![b]\!]$ in the state we get after executing s once in state σ.

- $\mathcal{B}[\![b]\!]\sigma' = tt$
  The last step in the construction of T was to use the first `repeat` rule. Thus, we know that $\langle s, \sigma\rangle \rightarrow \sigma'$ holds. Furthermore, we know that for all states σ' transition $\langle \texttt{skip}, \sigma'\rangle \rightarrow \sigma'$ holds. Using these two transitions and condition $\mathcal{B}[\![b]\!]\sigma' = tt$ we can construct derivation tree:

$$\frac{\langle s, \sigma\rangle \rightarrow \sigma', \quad \dfrac{\langle \texttt{skip}, \sigma'\rangle \rightarrow \sigma'}{\langle \texttt{if}\ b\ \texttt{then skip else repeat}\ s\ \texttt{until}\ b\ \texttt{end}, \sigma'\rangle \rightarrow \sigma'}\ \mathcal{B}[\![b]\!]\sigma' = tt}{\langle s; \texttt{if}\ b\ \texttt{then skip else repeat}\ s\ \texttt{until}\ b\ \texttt{end}, \sigma\rangle \rightarrow \sigma'}$$

- $\mathcal{B}[\![b]\!]\sigma'' = ff$
  The last step in the construction of T was to use the second `repeat` rule. Thus, we know that $\langle s, \sigma\rangle \rightarrow \sigma''$ and $\langle \texttt{repeat}\ s\ \texttt{until}\ b, \sigma''\rangle \rightarrow \sigma'$ hold. Using these two transitions and condition $\mathcal{B}[\![b]\!]\sigma'' = ff$ we can construct derivation tree:

$$\frac{\langle s, \sigma\rangle \rightarrow \sigma'', \quad \dfrac{\langle \texttt{repeat}\ s\ \texttt{until}\ b, \sigma''\rangle \rightarrow \sigma'}{\langle \texttt{if}\ b\ \texttt{then skip else repeat}\ s\ \texttt{until}\ b\ \texttt{end}, \sigma''\rangle \rightarrow \sigma'}\ \mathcal{B}[\![b]\!]\sigma'' = ff}{\langle s; \texttt{if}\ b\ \texttt{then skip else repeat}\ s\ \texttt{until}\ b\ \texttt{end}, \sigma\rangle \rightarrow \sigma'}$$

2. Direction $\Longleftarrow$: we assume there is a derivation tree T for

$$\langle s\texttt{;if}\ b\ \texttt{then skip else repeat}\ s\ \texttt{until}\ b\ \texttt{end}, \sigma\rangle \rightarrow \sigma''$$

and have to show that there exists one for

$$\langle \texttt{repeat}\ s\ \texttt{until}\ b, \sigma\rangle \to \sigma''.$$

The last step in the construction of T was to use the composition rule:

$$\frac{\langle s, \sigma\rangle \to \sigma', \quad \langle \texttt{if}\ b\ \texttt{then skip else repeat}\ s\ \texttt{until}\ b\ \texttt{end}, \sigma'\rangle \to \sigma''}{\langle s; \texttt{if}\ b\ \texttt{then skip else repeat}\ s\ \texttt{until}\ b\ \texttt{end}, \sigma\rangle \to \sigma''} \quad (*)$$

We make a case split on the value of $\mathcal{B}[\![b]\!]\sigma'$.

- $\mathcal{B}[\![b]\!]\sigma' = tt$
  Using the left-hand side premise of (*), we can use the first `repeat` rule to construct derivation tree T_1:

  $$\frac{\langle s, \sigma\rangle \to \sigma'}{\langle \texttt{repeat}\ s\ \texttt{until}\ b, \sigma\rangle \to \sigma'} \quad \mathcal{B}[\![b]\!]\sigma' = tt$$

  Since $\mathcal{B}[\![b]\!]\sigma' = tt$, from the right-hand side premise of (*) we can deduce $\langle \texttt{skip}, \sigma'\rangle \to \sigma''$, thus we know that $\sigma' = \sigma''$. Using this result and the root of T_1 we get $\langle \texttt{repeat}\ s\ \texttt{until}\ b, \sigma\rangle \to \sigma''$.

- $\mathcal{B}[\![b]\!]\sigma' = ff$
  From the right-hand side premise of (*) we can deduce $\langle \texttt{repeat}\ s\ \texttt{until}\ b, \sigma'\rangle \to \sigma''$. Using this result and the left-hand side premise of (*) we can use the second `repeat` rule to construct derivation tree:

  $$\frac{\langle s, \sigma\rangle \to \sigma', \quad \langle \texttt{repeat}\ s\ \texttt{until}\ b, \sigma'\rangle \to \sigma''}{\langle \texttt{repeat}\ s\ \texttt{until}\ b, \sigma\rangle \to \sigma''} \quad \mathcal{B}[\![b]\!]\sigma' = ff$$