Homework Assignment 3 Solution

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Problem 1

We shall prove all the properties with a Kripke model $\mathfrak{M} = (W, (\mathcal{K}_i)_i, (\mathcal{D}_G)_G, \pi)$ and an arbitrary world $w \in W$. (An interpreted system $\mathcal{I}$ and an arbitrary point (r, m) in it can be considered as a special case and are sufficient, too.)

(a) Assuming $\mathfrak{M}, w \Vdash D_G\varphi \wedge D_G(\varphi \supset \psi)$, we know that $\mathfrak{M}, w' \Vdash \varphi$ and $\mathfrak{M}, w' \Vdash \varphi \supset \psi$ for all points $w' \in \mathcal{D}_G(w)$. These give $\mathfrak{M}, w' \Vdash \psi$, thus $\mathfrak{M}, w \Vdash D_G\psi$. So $(D_G\varphi \wedge D_G(\varphi \supset \psi)) \supset D_G\psi$ is valid.

(b) Assuming $\mathfrak{M}, w \Vdash D_G\varphi$, we know that $\mathfrak{M}, w' \Vdash \varphi$ for all points $w' \in \mathcal{D}_G(w)$. Since $\mathcal{K}_i$ are all reflexive, $\mathcal{D}_G = \cap_{i \in G}\mathcal{K}_i$ is also reflexive. Thus $w \in \mathcal{D}_G(w)$, and $\mathfrak{M}, w \Vdash \varphi$ also holds. So $D_G\varphi \supset \varphi$ is valid.

(c) Assuming $\mathfrak{M}, w \Vdash D_G\varphi$, we know that $\mathfrak{M}, w' \Vdash \varphi$ for all points $w' \in \mathcal{D}_G(w)$.

Now we want to prove that for any $w'' \in \mathcal{D}_G(w')$, it is still true that $\mathfrak{M}, w'' \Vdash \varphi$. This is because that $\mathcal{K}_i$ are all transitive, $\mathcal{D}_G$ is also transitive, and $w'' \in \mathcal{D}_G(w)$, which implies $\mathfrak{M}, w'' \Vdash \varphi$.

So that $\mathfrak{M}, w' \Vdash D_G\varphi$, $\mathfrak{M}, w \Vdash D_G D_G\varphi$, and $D_G\varphi \supset D_G D_G\varphi$ is valid.

(d) Assuming $\mathfrak{M}, w \Vdash \neg D_G\varphi$, we know that there exists a point $w' \in \mathcal{D}_G(w)$ such that $\mathfrak{M}, w' \nVdash \varphi$.

Then since $\mathcal{K}_i$ are all symmetric and transitive, $\mathcal{D}_G$ is also symmetric and transitive. Which means that for all $w'' \in D_G(w)$, $w \in D_G(w'')$, and finally $w' \in D_G(w'')$ by transitivity.

This implies that $\mathfrak{M}, w'' \Vdash \neg D_G\varphi$ since w' is a counterexample. Finally we get $\mathfrak{M}, w \Vdash D_G\neg D_G\varphi$ and $\neg D_G\varphi \supset D_G\neg D_G\varphi$ is valid.

(e) Since $\mathcal{D}_{\{i\}}$ is defined as $\cap_{\{j \in \{i\}\}}\mathcal{K}_j = \mathcal{K}_i$, $D_{\{i\}}\varphi$ is identical to $K_i\varphi$.

(f) From $G \subseteq G'$ we have

$$\mathcal{D}_G = \bigcap_{i \in G}\mathcal{K}_i \supseteq \bigcap_{i \in G'}\mathcal{K}_i = \mathcal{D}_{G'}$$

Assuming $\mathfrak{M}, w \Vdash D_G\varphi$, we know that $\mathfrak{M}, w' \Vdash \varphi$ for all points $w' \in \mathcal{D}_G(w)$. Then for every points $w'' \in \mathcal{D}_{G'} \subseteq \mathcal{D}_G$, $\mathfrak{M}, w'' \Vdash \varphi$, which implies that $\mathfrak{M}, w \Vdash D_{G'}\varphi$, and $D_G\varphi \supset D_{G'}\varphi$ is valid given that $G \subseteq G'$.

Problem 2

(a) Given $\mathfrak{M} = (W, (R_a)_{a\in\Pi_0}, \pi)$, for every $w \in W$, assume that $\mathfrak{M}, w \Vdash \langle\alpha\rangle\varphi$. There is a $u \in W$ such that $(w,u) \in R_\alpha$ and $\mathfrak{M}, u \Vdash \varphi$. Then

$$\begin{aligned}(w,u) &\in R_{\alpha^-}\\ (w,w) &\in R_\alpha \circ R_{\alpha^-}\\ (w,u) &\in R_\alpha \circ R_{\alpha^-} \circ R_\alpha = R_{\alpha\alpha^-\alpha}\end{aligned}$$

and $\mathfrak{M}, w \Vdash \langle\alpha\alpha^-\alpha\rangle\varphi$. Which means $\langle\alpha\rangle\varphi \supset \langle\alpha\alpha^-\alpha\rangle\varphi$.

(b) $\mathfrak{M}, w \Vdash \langle\alpha^*\alpha\rangle\varphi$ iff there exists $u \in W$ such that $(w,u) \in R_\alpha$, $\mathfrak{M}, u \Vdash \varphi$. Assume that it is v_n such that $(w, v_n) \in R^n_\alpha$ and $(v_n, u) \in R^n_\alpha$.

If $n = 0$ the implication is obvious. Otherwise, let $v_i \in W$, $0 \le i < n$, $v_0 = w$, and $(v_i, v_{i+1}) \in R_\alpha$. Then $(w, v_1) \in R_\alpha$ and $(v_1, u) \in R^n_\alpha$, which imply that $\mathfrak{M}, w \Vdash \langle\alpha\alpha^*\rangle\varphi$.

And the another direction is the same.

(c) This is almost the same as (b), except that the v_i are defined on $0 \le i \le 2n$ with $(w, v_0) \in R_a$, $v_{2n} = u$, $(v_{2j}, v_{2j+1}) \in R_b$, and $(v_{2j+1}, v_{2j+2}) \in R_a$ for $0 \le j < n$. Then $(w, v_{2n-1}) \in R_{(ab)^*}$, $(v_{2n-1}, u) \in R_a$ and $\langle a(ab)^*\rangle\varphi \equiv \langle(ab)^*a\rangle\varphi$.