Homework Assignment 1 Solution

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Problem 1

Choose

$$W = \{w\} \quad R = \emptyset \quad \pi(p) = \emptyset \text{for all } p$$

Then w has no successors, thus $\mathfrak{M}, w \Vdash \Box p$ holds for all p, and so does $\mathfrak{M}, w \Vdash \Box\neg p$. These yield $\mathfrak{M}, w \Vdash \Box p \wedge \Box\neg p$.

Problem 2

$$\begin{aligned}
& \mathfrak{M} \Vdash \Box(\Box\varphi \supset \varphi) \\
\text{iff} \quad & \forall w \in W, \mathfrak{M}, w \Vdash \Box(\Box\varphi \supset \varphi) \\
\text{iff} \quad & \forall w, u \in W, (w, u) \in R, \mathfrak{M}, u \Vdash \Box\varphi \supset \varphi \\
\text{iff} \quad & \forall w, u \in W, (w, u) \in R, \text{given } \mathfrak{M}, u \Vdash \Box\varphi \text{ it can be shown that } \mathfrak{M}, u \Vdash \varphi
\end{aligned}$$

But by definition, for such u, $(u, u) \in R$, and $\mathfrak{M}, u \Vdash \Box\varphi$ implies $\mathfrak{M}, u \Vdash \varphi$, then we are done.

Problem 3(a)

Choose

$$W = \{w_1, w_2\} \quad R = \{(w_1, w_2)\} \quad \pi(p) = \{w_1\}$$

then $\mathfrak{M}, w_1 \Vdash p$ but since $\mathfrak{M}, w_2 \nVdash p$, $\mathfrak{M}, w_1 \nVdash \Box p$.

If $\{p\} \Vdash_{\mathsf{C}} \Box p$, every model $\mathfrak{M}$ and world w in it with $\mathfrak{M}, w \Vdash p$ must implies $\mathfrak{M}, w \nVdash \Box p$. But the setting in the previous paragraph is a counterexample, thus $\{p\} \nVdash_{\mathsf{C}} \Box p$.

Problem 3(b)

To prove that $\{p\} \Vdash^g_{\mathsf{C}} \Box p$, assume $\mathfrak{M} \in \mathsf{C}$ such that $\mathfrak{M} \Vdash p$, we shall prove that $\mathfrak{M} \Vdash \Box p$. But this is obvious since every worlds in $\mathfrak{M}$ satisfies p, every successors of every worlds also does, which means every worlds in $\mathfrak{M}$ satisfies $\Box p$.

Problem 4

Let

$$\begin{array}{rcccl} \mathfrak{M} & = & (W, R, \pi) & = & (\mathbb{Z}, \{(i, i+1)\}_{i \in \mathbb{Z}}, \pi) \\ \mathfrak{M} & = & (W', R', \pi') & = & (\mathbb{N}, \{(i, i+1)\}_{i \in \mathbb{N}}, \pi|_{\mathbb{N}}) \end{array}$$

where π is chosen so that $w = 0$ is the only world in w such that $\mathfrak{M}, w \Vdash \varphi$.

Obviously $\mathfrak{M}' \rightarrowtail \mathfrak{M}$. So for every $w \in W'$, $\mathfrak{M}, w \Vdash \varphi \Leftrightarrow \mathfrak{M}', w \Vdash \varphi$. Suppose $\overleftarrow{\Diamond}\, \varphi$ is a wff, choose $w = 1$. We have $\mathfrak{M}, w \Vdash \overleftarrow{\Diamond}\, \varphi$ since $\mathfrak{M}, 0 \Vdash \varphi$, but in $\mathfrak{M}'$, $\mathfrak{M}', w \nVdash \overleftarrow{\Diamond}\, \varphi$ since there is no u such that $(u, w) \in R'$.

So $\overleftarrow{\Diamond}\, \varphi$ is not definable.

Problem 5

Let Z be the bisimulation such that for every $w \in W$, $(w, f(w)) \in Z$. Check:

- if wZw' then $w \in \pi(p)$ iff $w' \in \pi'(p)$ for every p:
  By definition, $w \in \pi(p)$ iff $f(w) \in \pi'(p)$ for every p.
- if wZw' and $(w, u) \in R$ then there exists $w' \in W'$ such that uZu' and $(w', u') \in R'$:
  By definition, $(w, u) \in R$ implies $(f(w), f(u)) \in R'$, so $u' = f(u)$ satisfies.
- if wZw' and $(w', u') \in R'$ then there exists $u \in W$ such that uZu' and $(w, u) \in R$:
  By definition, if $(f(w), u') \in R'$ then there exists $u \in W$ such that $(w, u) \in R$ and $f(u) = u'$. Such u satisfies.