Suggested Solutions to Homework Assignment #1

[Compiled on July 7, 2009]

Problems

1. (20 Points) Prove the following tautological implication in sentential logic. P, Q, and R are sentence symbols.

   - $\{P \to (Q \wedge R)\} \vDash (P \to Q) \wedge (P \to R)$

     Solution.
     The following truth table shows that every truth assignment satisfying $P \to (Q \wedge R)$ also satisfies $(P \to Q) \wedge (P \to R)$.

P	Q	R	$Q \wedge R$	$P \to Q$	$P \to R$	$P \to (Q \wedge R)$	$(P \to Q) \wedge (P \to R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F
T	F	T	F	F	T	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

   - $\{(P \wedge Q) \to R\} \vDash (P \to R) \vee (Q \to R)$

     Solution.
     The following truth table shows that every truth assignment satisfying $(P \wedge Q) \to R$ also satisfies $(P \to R) \vee (Q \to R)$.

P	Q	R	$P \wedge Q$	$P \to R$	$Q \to R$	$(P \wedge Q) \to R$	$(P \to R) \vee (Q \to R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	F	T	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	F	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

2. (20 Points) Assume that a set Σ of wffs is finitely satisfiable and α is a wff. Show that either $\Sigma \cup \{\alpha\}$ or $\Sigma \cup \{\neg\alpha\}$ is finitely satisfiable.

   Solution.
   We prove this by contradiction. Assume both $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$ are not finitely

satisfiable. Since Σ is finitely satisfiable, there are finite sets $\Sigma_1 \subseteq \Sigma$ and $\Sigma_2 \subseteq \Sigma$ such that (1) both $\Sigma_1 \cup \{\alpha\}$ and $\Sigma_1 \cup \{\neg\alpha\}$ are not satisfiable and (2) $\Sigma_1 \cup \Sigma_2$ is satisfiable. Since $\Sigma_1 \cup \{\alpha\}$ is not satisfiable, we have $\Sigma_1 \vDash \neg\alpha$. Similarly, $\Sigma_2 \vDash \alpha$. Thus $\Sigma_1 \cup \Sigma_2 \vDash \alpha$ and $\Sigma_1 \cup \Sigma_2 \vDash \neg\alpha$. By Lemma 0.1 below, $\Sigma_1 \cup \Sigma_2$ is not satisfiable, which is a contradiction.

Lemma 0.1. *Given a set Σ of wffs and a wff α, if $\Sigma \vDash \alpha$ and $\Sigma \vDash \neg\alpha$, then Σ is not satisfiable.*

Proof. If Σ is satisfiable, then there is a truth assignment v such that v satisfies every member of Σ. Since $\Sigma \vDash \alpha$ and $\Sigma \vDash \neg\alpha$, we have $\bar{v}(\alpha) = \mathsf{T}$ and $\bar{v}(\neg\alpha) = \mathsf{T}$. By definition, $\bar{v}(\neg\alpha) = \mathsf{T}$ implies that $\bar{v}(\alpha) = \mathsf{F}$, which is a contradiction. □

3. (20 Points) Consider the structure $\mathfrak{N} = (\mathbb{N}, S, 0, +, *)$. Write a first-order logic formula to define the set of odd numbers.

   Solution.
   $\exists v_2 (v_1 \approx SS0 * v_2 + 1)$

4. (20 Points) Show that $(\mathbb{N}, +_{\mathbb{N}})$ and $(\mathbb{Z}, +_{\mathbb{Z}})$ are not elementarily equivalent by giving a sentence true in one but false in the other.

   Solution.
   The sentence $\forall x \forall y \exists z (x = y + z)$ is valid in $(\mathbb{Z}, +_{\mathbb{Z}})$ but not valid in $(\mathbb{N}, +_{\mathbb{N}})$.

5. (20 Points) Find a deduction (from $\varnothing$) for each of the following formulae.

   - $\exists x(\alpha \wedge \beta) \to \exists x\alpha \wedge \exists x\beta$ *Solution.*

     (a) $\forall x(\neg\alpha \to \alpha \to \neg\beta)$ (tautology)
     (b) $\forall x(\neg\alpha \to \alpha \to \neg\beta) \to \forall x\neg\alpha \to \forall x(\alpha \to \neg\beta)$ (axiom group 3)
     (c) $\forall x\neg\alpha \to \forall x(\alpha \to \neg\beta)$ (5a, 5b, modus ponens)
     (d) $\forall x((\alpha \to \neg\beta) \to \neg\neg(\alpha \to \neg\beta))$ (tautology)
     (e) $\forall x((\alpha \to \neg\beta) \to \neg\neg(\alpha \to \neg\beta)) \to \forall x(\alpha \to \neg\beta) \to \forall x\neg\neg(\alpha \to \neg\beta)$ (axiom group 3)
     (f) $\forall x(\alpha \to \neg\beta) \to \forall x\neg\neg(\alpha \to \neg\beta)$ (5d, 5e, modus ponens)
     (g) $(\forall x\neg\alpha \to \forall x(\alpha \to \neg\beta)) \to (\forall x(\alpha \to \neg\beta) \to \forall x\neg\neg(\alpha \to \neg\beta)) \to (\forall x\neg\alpha \to \forall x\neg\neg(\alpha \to \neg\beta))$ (tautology)
     (h) $(\forall x(\alpha \to \neg\beta) \to \forall x\neg\neg(\alpha \to \neg\beta)) \to (\forall x\neg\alpha \to \forall x\neg\neg(\alpha \to \neg\beta))$ (5c, 5g, modus ponens)
     (i) $\forall x\neg\alpha \to \forall x\neg\neg(\alpha \to \neg\beta)$ (5f, 5h, modus ponens)
     (j) $(\forall x\neg\alpha \to \forall x\neg\neg(\alpha \to \neg\beta)) \to \neg\forall\neg\neg(\alpha \to \neg\beta) \to \neg\forall x\neg\alpha$ (tautology)
     (k) $\exists(\alpha \wedge \beta) \to \exists x\alpha$ (5i, 5j, modus ponens)
     (l) $\forall x(\neg\beta \to \alpha \to \neg\beta)$ (tautology)
     (m) $\forall x(\neg\beta \to \alpha \to \neg\beta) \to \forall x\neg\beta \to \forall x(\alpha \to \neg\beta)$ (axiom group 3)
     (n) $\forall x\neg\beta \to \forall x(\alpha \to \neg\beta)$ (5l, 5m, modus ponens)

(o) $(\forall x\neg\beta \to \forall x(\alpha \to \neg\beta)) \to (\forall x(\alpha \to \neg\beta) \to \forall x\neg\neg(\alpha \to \neg\beta)) \to (\forall x\neg\beta \to \forall x\neg\neg(\alpha \to \neg\beta))$ (tautology)

(p) $(\forall x(\alpha \to \neg\beta) \to \forall x\neg\neg(\alpha \to \neg\beta)) \to (\forall x\neg\beta \to \forall x\neg\neg(\alpha \to \neg\beta))$ (5n, 5o, modus ponens)

(q) $\forall x\neg\beta \to \forall x\neg\neg(\alpha \to \neg\beta)$ (5f, 5p, modus ponens)

(r) $(\forall x\neg\beta \to \forall x\neg\neg(\alpha \to \neg\beta)) \to \neg\forall\neg\neg(\alpha \to \neg\beta) \to \neg\forall x\neg\beta$ (tautology)

(s) $\exists(\alpha \wedge \beta) \to \exists x\beta$ (5q, 5r, modus ponens)

(t) $(\exists x(\alpha \wedge \beta) \to \exists x\alpha) \to (\exists x(\alpha \wedge \beta) \to \exists x\beta) \to (\exists x(\alpha \wedge \beta) \to \exists x\alpha \wedge \exists x\beta)$ (tautology)

(u) $(\exists x(\alpha \wedge \beta) \to \exists x\beta) \to (\exists x(\alpha \wedge \beta) \to \exists x\alpha \wedge \exists x\beta)$ (5k, 5t, modus ponens)

(v) $\exists x(\alpha \wedge \beta) \to \exists x\alpha \wedge \exists x\beta$ (5s, 5u, modus ponens)

- $Py \to \forall x(x \approx y \to Px)$ *Solution.*

(a) $\forall x((x \approx y \to y \approx x) \to (y \approx x \to Py \to Px) \to x \approx y \to Py \to Px)$ (tautology)

(b) $\forall x((x \approx y \to y \approx x) \to (y \approx x \to Py \to Px) \to x \approx y \to Py \to Px) \to \forall x(x \approx y \to y \approx x) \to \forall x((y \approx x \to Py \to Px) \to x \approx y \to Py \to Px)$ (axiom group 3)

(c) $\forall x(x \approx y \to y \approx x) \to \forall x((y \approx x \to Py \to Px) \to x \approx y \to Py \to Px)$ (5a, 5b, modus ponens)

(d) $\forall x(x \approx y \to x \approx x \to y \approx x)$ (axiom group 6)

(e) $\forall x((x \approx y \to x \approx x \to y \approx x) \to (x \approx x \to x \approx y \to y \approx x))$ (tautology)

(f) $\forall x((x \approx y \to x \approx x \to y \approx x) \to (x \approx x \to x \approx y \to y \approx x)) \to \forall x(x \approx y \to x \approx x \to y \approx x) \to \forall x(x \approx x \to x \approx y \to y \approx x)$ (axiom group 3)

(g) $\forall x(x \approx y \to x \approx x \to y \approx x) \to \forall x(x \approx x \to x \approx y \to y \approx x)$ (5e, 5f, modus ponens)

(h) $\forall x(x \approx x \to x \approx y \to y \approx x)$ (5d, 5g, modus ponens)

(i) $\forall x(x \approx x \to x \approx y \to y \approx x) \to \forall x(x \approx x) \to \forall x(x \approx y \to y \approx x)$ (axiom group 3)

(j) $\forall x(x \approx x) \to \forall x(x \approx y \to y \approx x)$ (5h, 5i, modus ponens)

(k) $\forall x(x \approx x)$ (axiom group 5)

(l) $\forall x(x \approx y \to y \approx x)$ (5j, 5k, modus ponens)

(m) $\forall x((y \approx x \to Py \to Px) \to x \approx y \to Py \to Px)$ (5c, 5l, modus ponens)

(n) $\forall x((y \approx x \to Py \to Px) \to x \approx y \to Py \to Px) \to \forall x(y \approx x \to Py \to Px) \to \forall x(x \approx y \to Py \to Px)$ (axiom group 3)

(o) $\forall x(y \approx x \to Py \to Px) \to \forall x(x \approx y \to Py \to Px)$ (5m, 5n, modus ponens)

(p) $\forall x(y \approx x \to Py \to Px)$ (axiom group 6)

(q) $\forall x(x \approx y \to Py \to Px)$ (5o, 5p, modus ponens)

(r) $\forall x((x \approx y \to Py \to Px) \to Py \to x \approx y \to Px)$ (tautology)

(s) $\forall x((x \approx y \to Py \to Px) \to Py \to x \approx y \to Px) \to \forall x(x \approx y \to Py \to Px) \to \forall x(Py \to x \approx y \to Px)$ (axiom group 3)

(t) $\forall x(x \approx y \to Py \to Px) \to \forall x(Py \to x \approx y \to Px)$ (5r, 5s, modus ponens)

(u) $\forall x(Py \to x \approx y \to Px)$ (5q, 5t, modus ponens)

(v) $\forall x(Py \to x \approx y \to Px) \to \forall x Py \to \forall x(x \approx y \to Px)$ (axiom group 3)

(w) $\forall x Py \to \forall x(x \approx y \to Px)$ (5u, 5v, modus ponens)

(x) $Py \to \forall x Py$ (axiom group 4)

(y) $(Py \to \forall x Py) \to (\forall x Py \to \forall x(x \approx y \to Px)) \to (Py \to \forall x(x \approx y \to Px))$ (tautology)

(z) $(\forall x Py \to \forall x(x \approx y \to Px)) \to (Py \to \forall x(x \approx y \to Px))$ (5x, 5y, modus ponens)

(aa) $Py \to \forall x(x \approx y \to Px)$ (5w, 5z, modus ponens)