Büchi Complementation

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Introduction

- Languages recognizable by (nondeterministic) Büchi automata are called ω-regular languages.
- The class of ω-regular languages is closed under intersection and complementation (and hence all boolean operations).
- Deterministic Büchi automata are strictly less expressive.
- The complement of a deterministic Büchi automaton may not be deterministic.

Introduction (cont.)

- While intersection is rather straightforward, complementation is much harder and still a current research topic.
- A complementation construction is also useful for checking language containment (and hence equivalence) between two automata:

$$L(A) \subseteq L(B) \equiv L(A) \cap L(\overline{B}) = \phi.$$

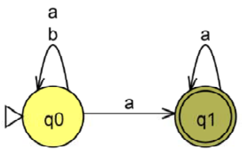
- The language containment test is essential in the automata-theoretic approach to model checking (more about this later …).

Complementation of an NFA

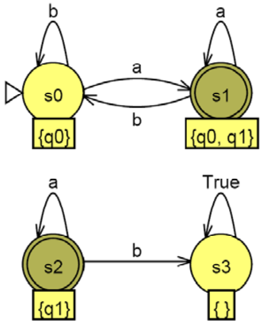
- Translate the given nondeterministic finite automaton (NFA) *N* into an equivalent deterministic finite automaton (DFA) *D* via the subset construction.
- Take the dual of *D* to get a DFA *D'* for the complement language.
- This works because languages recognizable by DFA's are closed under complementation.

Example of NFA Complementation

- L(*N*) = (a+b)*aa*, which equals (a+b)*a.
- An equivalent DFA *D* by the subset construction.



NFA *N*



DFA *D*

There are two unreachable states in *D*.

Subset Construction for Finite Words

- Formally, from NFA *N*=(S_N, Σ, δ_N, q_0, F_N), we construct an equivalent DFA *D*=(S_D, Σ, δ_D, {q_0}, F_D) as follows:
  - $S_D = 2^{S_N}$
  - $\delta_D(S,a) = \bigcup_{s \in S} \delta_N(s,a)$
  - $F_D = \{S \in S_D \mid S \cap F_N \neq \phi\}$

ω-Automata

- ω-automata are finite automata on infinite words.
- Büchi automata are one type of ω-automata.
- Formally, a (nondeterministic) ω-automaton *B* is represented as a five-tuple *B*=(Σ, S, s_0, δ, *Acc*):
  - Σ: a finite alphabet (set of symbols)
  - S: a finite set of states (or locations)
  - $s_0 \in S$: the initial state
  - δ: $S \times \Sigma \rightarrow 2^S$
  - *Acc*: the acceptance condition
- When δ is actually a function from $S \times \Sigma$ to S, the automaton is said to be *deterministic*.

Runs and Languages of ω-Automata

- A *run* of an ω-automaton *B* on a word $w = w_1 w_2 \ldots$ is an infinite sequence of states $s_0 s_1 \ldots \in S^\omega$ such that for all $j \geq 0$ we have $s_{j+1} \in \delta(s_j, w_{j+1})$.
- For a run *r*, let Inf(*r*) denote the set of states that occur infinitely many times in *r*.
- A word *w* is *accepted* by *B* if there exists an *accepting* run of *B* on *w* that satisfies the acceptance condition.
- The language of *B*, denoted *L(B)*, is the set of all words accepted by *B*.

Büchi and Other ω-Automata

- Büchi automata:

  $$Acc = F \subseteq S.$$

  A run *r* is accepting iff $\mathrm{Inf}(r) \cap F \neq \phi$.

- Parity automata:

  $$Acc = \{F_0, F_1, \ldots, F_k\}, F_i \subseteq S.$$

  A run *r* is accepting iff the smallest *i* such that $\mathrm{Inf}(r) \cap F_i \neq \phi$ is even.

Büchi and Other ω-Automata (cont.)

- Rabin automata:

  $$Acc = \{(E_1, F_1), (E_2, F_2), \ldots, (E_k, F_k)\}, E_i, F_i \subseteq S.$$

  A run *r* is accepting iff for some *i*, $\mathrm{Inf}(r) \cap E_i = \phi$ and $\mathrm{Inf}(r) \cap F_i \neq \phi$.

- Streett automata:

  $$Acc = \{(E_1, F_1), (E_2, F_2), \ldots, (E_k, F_k)\}, E_i, F_i \subseteq S.$$

  A run *r* is accepting iff for all *i*, $\mathrm{Inf}(r) \cap E_i \neq \phi$ or $\mathrm{Inf}(r) \cap F_i = \phi$.

- Rabin automata and Streett automata are the dual of each other.

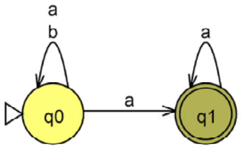
Convenient Acronyms

- DBW (or DBA): deterministic Büchi automata
- NBW: nondeterministic Büchi automata
- DPW: deterministic parity automata
- DRW: deterministic Rabin automata
- DSW: deterministic Streett automata
- etc.

Note: replace W with T, for tree automata.

An Example of Büchi Automaton

- *B* = ({a, b}, {q0, q1}, {q0}, T, {q1})
    - T(q0,a) = {q0, q1}
    - T(q0,b) = {q0}
    - T(q1,a) = {q1}
    - T(q1,b) = { }



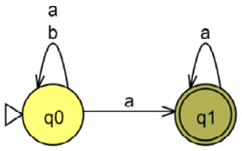
- Apparently, *B* is nondeterministic.
- *L*(*B*) = $(a+b)^*a^\omega$ (or "FG a" or "<>[]a").

Subset Construction for Infinite Words

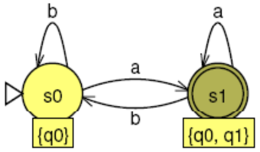
- If we use the subset construction to construct a DBW *D* from an NBW *N*, the two automata may not be language equivalent.
- By construction, the accepting states of the DBW *D* are those that contain an accepting state of the original NBW *N*.
- *D* may accept some words that are rejected by *N*, as shown by the following example.
- Thus, this method is not sound.

Naive Subset Construction

- NBW *N* defines the language: $(a+b)^*a^\omega$ ("eventually always a").



- *N* accepts words like $ababa^\omega$ and $bbba^\omega$.
- *N* rejects words like $(ab)^\omega$ and $bb(ba)^\omega$.
- A DBW *D* by the naive subset construction.



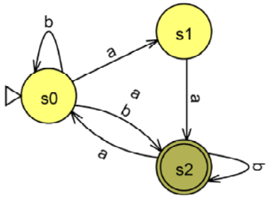
(unreachable states removed)

- *D* accepts every word that is accepted by *N*.
- However, *D* also accepts some words that are rejected by *N*, e.g., $(ab)^\omega$.

Another Subset Construction

- This subset construction keeps more detailed information of accepting states visited in a run.
- A state of *D* is called a **breakpoint** if the state does not contain any unmark state of *N*.
- The construction will mark an accepting state of *N* and every state that has a marked predecessor.
- A word *w* is accepted if *D* identifies **infinitely many breakpoints** while reading *w*.
- This does not work, either; see the example next.

Another Subset Construction (cont.)

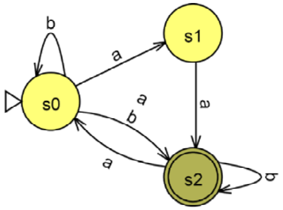


- This automaton accepts the input word a^ω.
- The constructed automaton also has a run on a^ω, which is accepting.

$$\boxed{\{s0\}} \xrightarrow{a} \boxed{\{s1,s2!\}} \xrightarrow{a} \overset{\text{breakpoint}}{\boxed{\{s0!,s2!\}}} \xrightarrow{a} \boxed{\{s0,s1,s2!\}} \xrightarrow{a} \boxed{\{s0!,s1,s2!\}}$$

$$\xrightarrow{a} \overset{\text{breakpoint}}{\boxed{\{s0!,s1!,s2!\}}} \xrightarrow{a} \boxed{\{s0,s1,s2!\}} \xrightarrow{a} \cdots$$

Another Subset Construction (cont.)



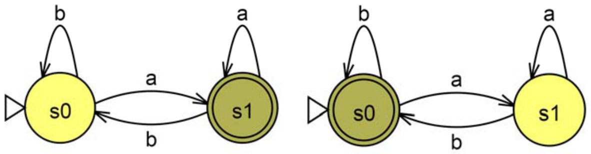
- This automaton also accepts the input word b^ω.
- However, the single run of the constructed automaton on b^ω is rejecting:

$$\boxed{\{s0\}} \xrightarrow{b} \boxed{\{s0,s2!\}} \xrightarrow{b} \boxed{\{s0,s2!\}} \xrightarrow{b} \boxed{\{s0,s2!\}} \xrightarrow{b} \cdots$$

- Therefore, this construction is incomplete, missing words that should be accepted.

Duality Does Not Apply

- If we take the dual of a given DBW D to get DBW D', then it is possible that $L(D) \cap L(D') \neq \phi$, e.g., $(ab)^\omega$.



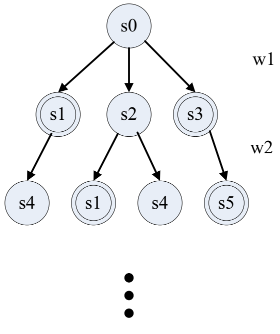
Note: DBW is not closed under complementation, e.g., $((a+b)^*a)^\omega$ (or GF a).

Muller-Schupp Construction

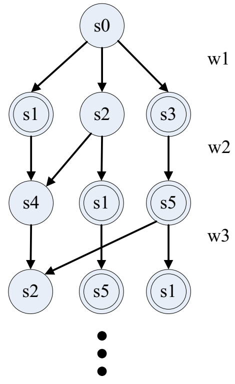
- We shall now study three constructions for Büchi complementation.
- Stages in Muller-Schupp construction:
  - NBW → DRW → (complete) DSW →NBW
  - The DSW is the complement of the DRW, by taking the dual view.
- The determinization part uses Muller-Schupp trees to construct the DRW.
- A Muller-Schupp tree (MS tree) is a finite strictly binary tree, which has precisely two children for each node except the leave nodes.

Run Trees vs. Run DAG's

- In Figure (a) is an example run tree r_w and in (b) is the corresponding run DAG r_d.



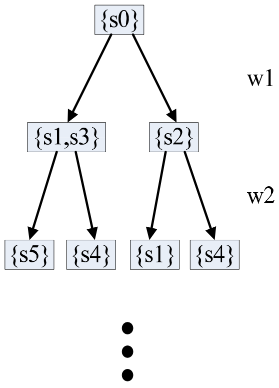
(a)



(b)

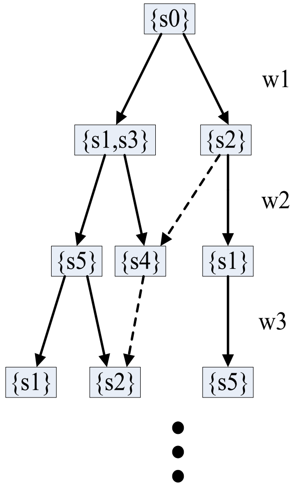
MS Trees

- In a run tree r_w, we partition the children of a node *v* into two classes, the left child which carries an accepting state and the right one which carries a non-accepting state.
- Let us refer to the new tree as t_1.
- Claim: r_w has an accepting path *iff* t_1 has a path branching left infinitely often.

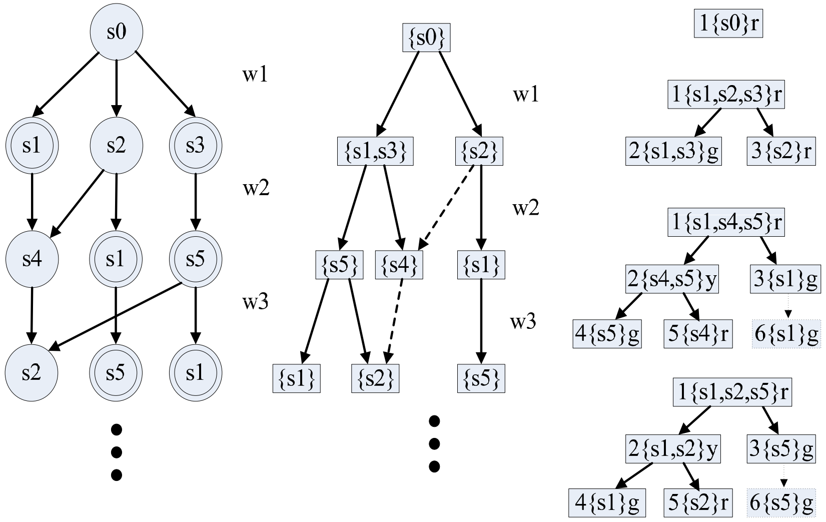


MS Trees (cont.)

- For every state *s* on each level in t_1, if we only keep the leftmost *s*, we obtain another new tree t_2
- Claim: t_1 has a path branching left infinitely often *iff* t_2 has a path branching left infinitely often.



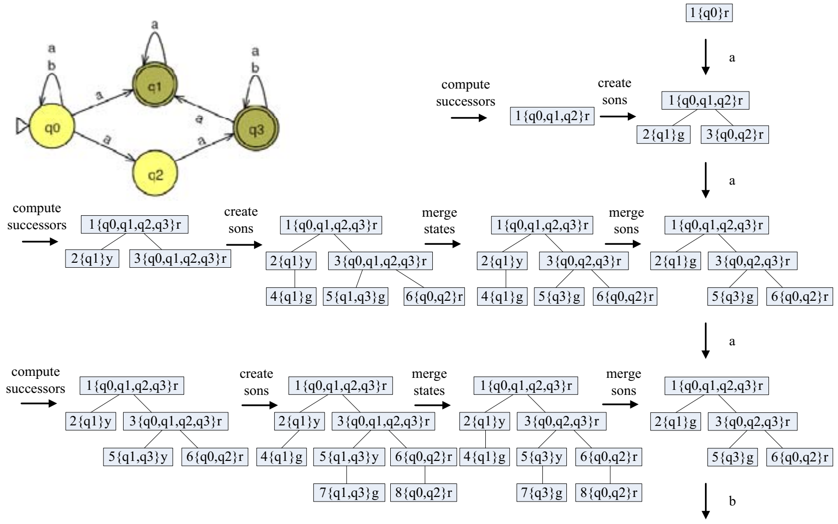
MS Trees (cont.)



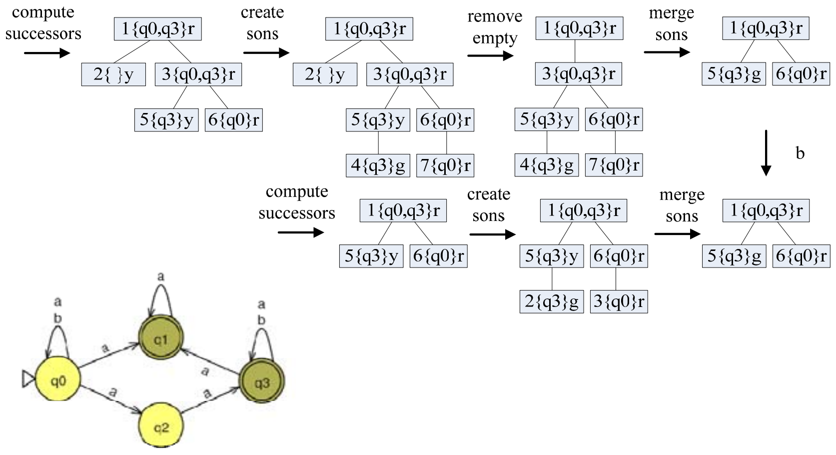
Three Colors for the Nodes

- Three colors are used to identify whether a node is accepting or not.
    - A node is *red* if the run path that the node represents has no accepting state.
    - A node is *yellow* if it has visited an accepting state before but it does not visit an accepting state in this step.
    - A node is *green* if it visits an accepting state in this step or it merges a green or yellow son.

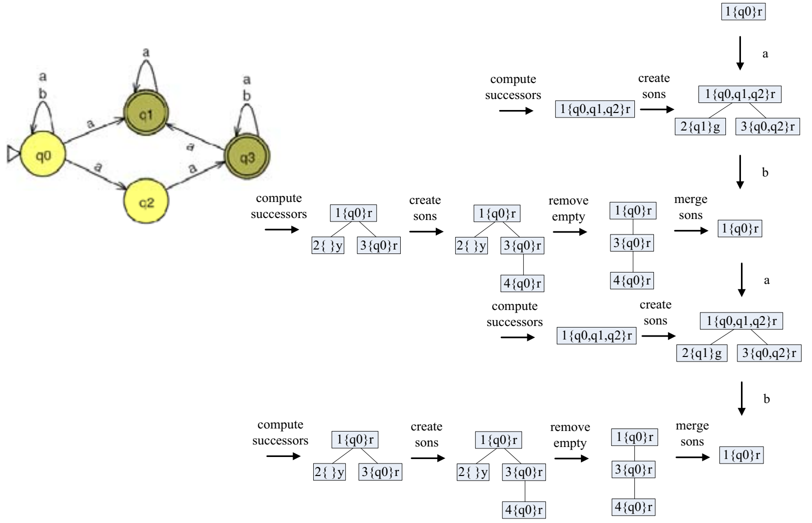
An Example of MS Construction



An Example of MS Construction (cont.)



An Example of Rejecting a Word



The Detail of Determinization

- Let $A = (\Sigma, S, s_0, \delta, F')$ be an NBW with n states.
- An equivalent DRW $D = (\Sigma, S', s_0', \delta', Acc)$:
  - S': a set of MS trees,
  - s_0': an initial MS tree with only one node numbered 1, which is labeled $\{s_0\}$ and colored red,
  - δ': a transition function which, given an input $a \in \Sigma$, transforms an MS tree using the steps described next.
  - $Acc = \{(E_1,F_1), (E_2,F_2), \ldots, (E_{4n},F_{4n})\}$:
    - E_i = the set of MS trees without node i.
    - F_i = the set of MS trees with green node i.

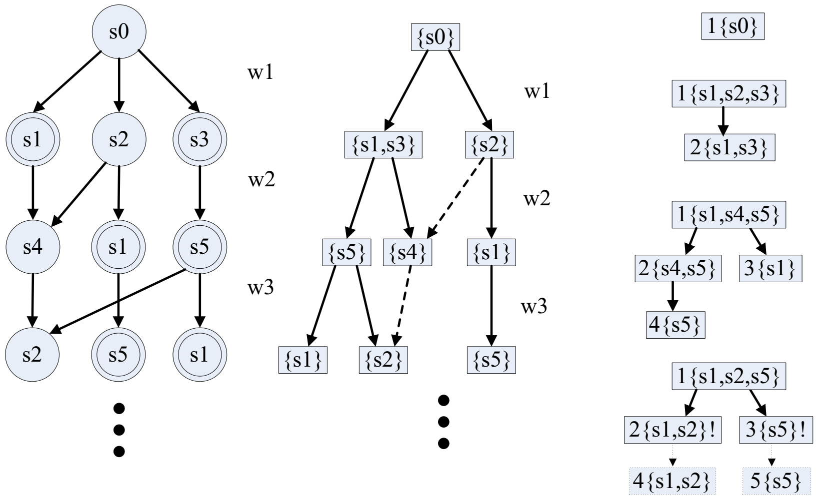
Detail of the Determinization (cont.)

- Steps to compute the next MS-tree state:
  - Change color green to yellow for every tree node.
  - Replace the label of every node with $\bigcup_{s \in L} \delta(s, a)$.
  - Create a left child with label $L \cap F$ and a right child with label $L \setminus F$.
  - Merge the same states into the leftmost one for each level in the tree.
  - Remove every node with an empty label.
  - Mark green every node that has only one child with color green or yellow.

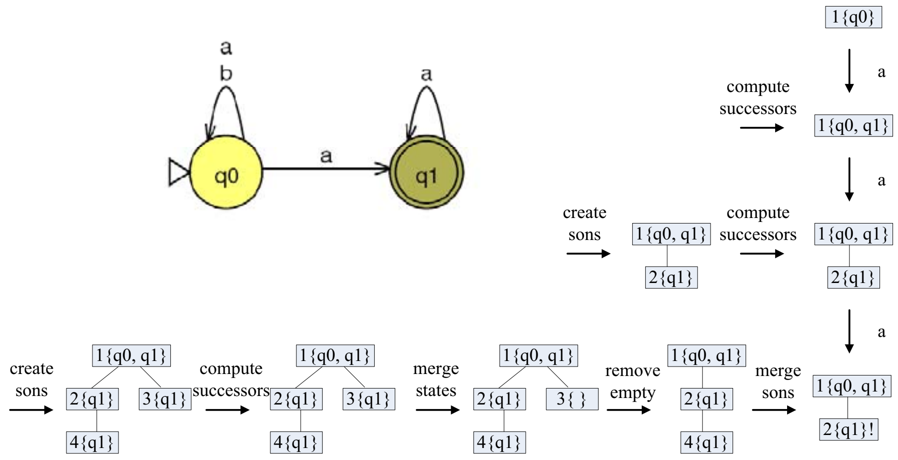
Safra's Construction

- Stages of the complementation:
  - NBW → DRW → (complement) DSW → NBW
- Safra trees are used to construct the DRW.
- Safra trees are labeled ordered trees.

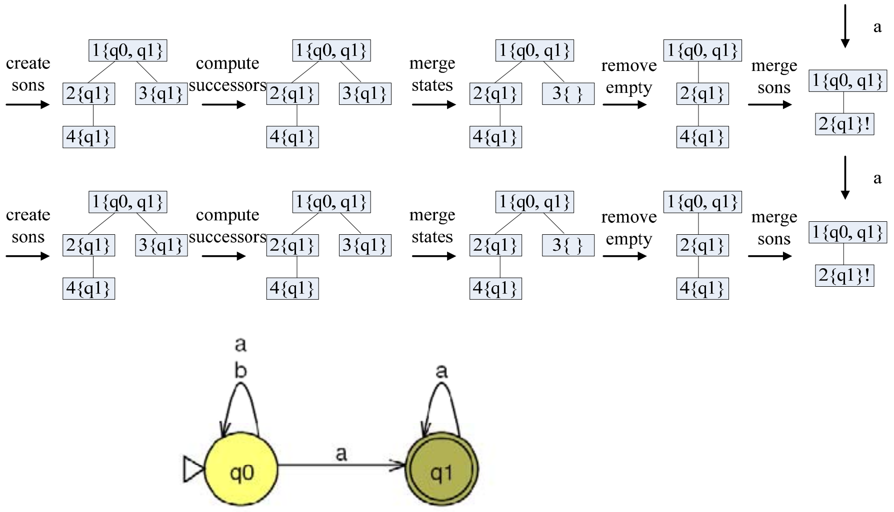
Safra Trees



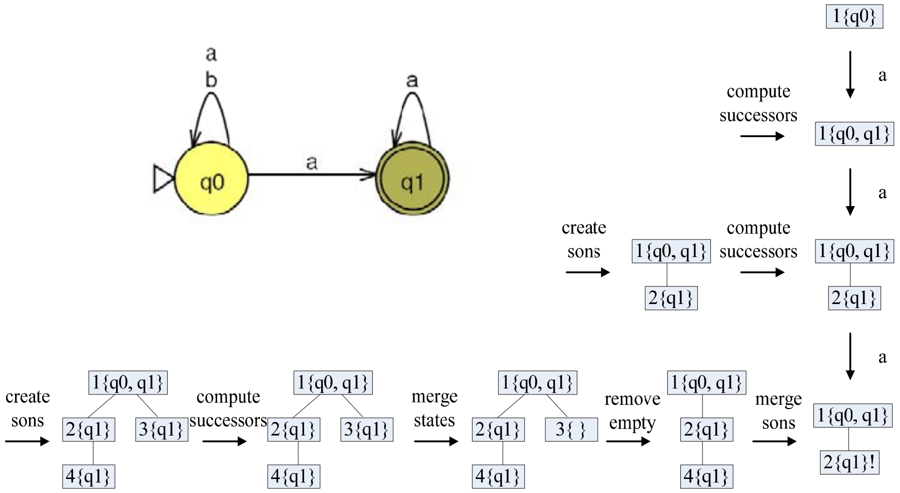
An Example of Construction



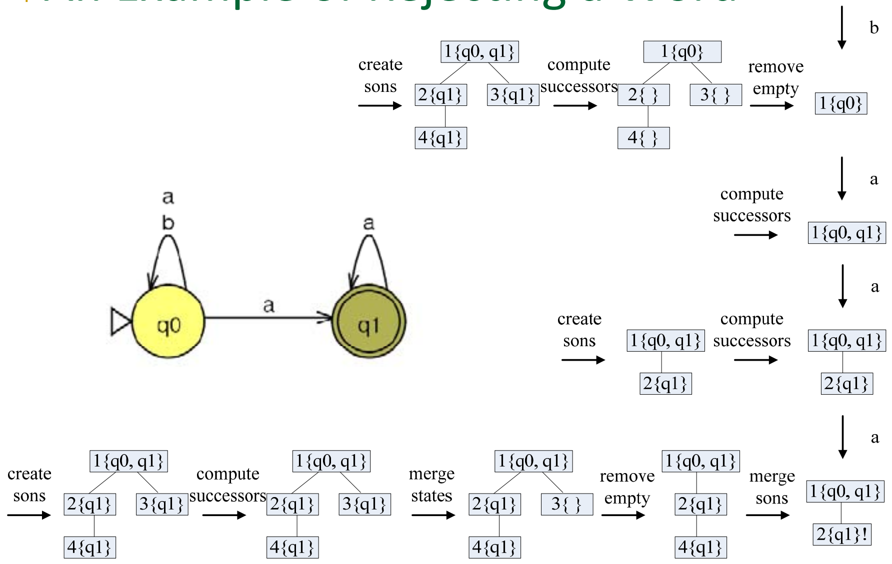
An Example of Construction (cont.)



An Example of Rejecting a Word



An Example of Rejecting a Word



Detail of the Determinization

- Let $A = (\Sigma, S, s_0, \delta, F)$ be an NBW with n states.
- An equivalent DRW $D = (\Sigma, S', s_0', \delta', Acc')$:
  - S': a set of Safra trees,
  - s_0': an initial Safra tree with only one node numbered 1 which is labeled $\{s_0\}$,
  - δ': a transition function which, given an input $a \in \Sigma$, transforms a Safra tree using the steps described next,
  - $Acc' = \{(E_1,F_1),(E_2,F_2), \dots, (E_{2n},F_{2n})\}$:
    - E_i = the set of Safra trees without node *i*.
    - F_i = the set of Safra trees with marked node *i*.

Detail of the Determinization (cont.)

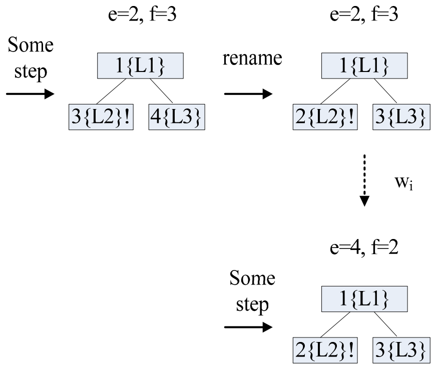
- Steps to compute the next Safra-tree state:
  - Remove the mark of every tree node.
  - Create a new child with label $L \cap F$.
  - Replace the label of every node with $\bigcup_{s \in L} \delta(s, a)$.
  - Merge the same states into the leftmost one for each level in the tree.
  - Remove every node with an empty label.
  - Mark every node whose label equals the union of the labels of its children and remove its children.

Safra-Piterman Construction

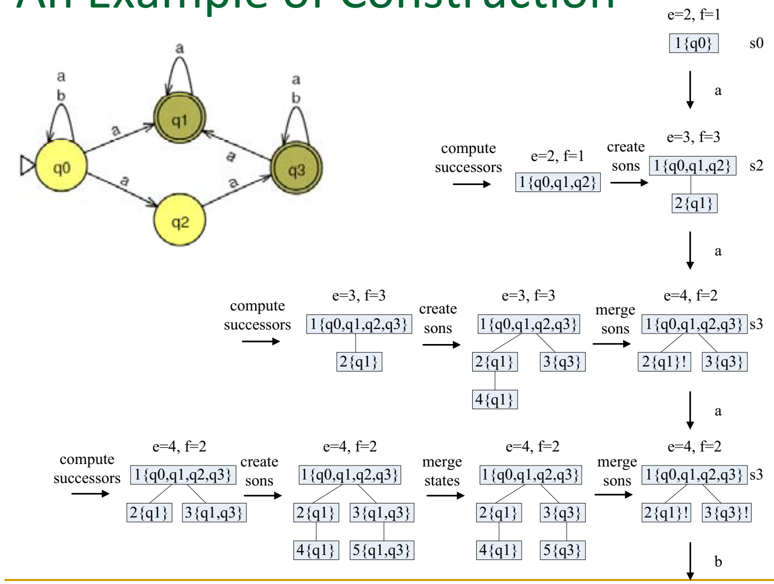
- Stages of the complementation:
  - NBW → DPW → (complement) DPW → NBW
- The determinization part uses compact Safra trees to construct the DPW.
- Compact Safra trees are Safra trees, but use two different kinds of techniques:
  - Dynamic names
  - Recording only the smallest marked name (called *f*) and removed name (called *e*)

Dynamic Names

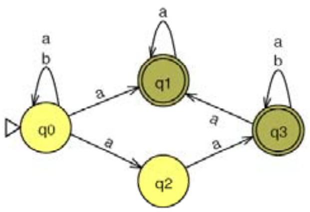
- The construction renames the tree at the final step and get a new tree.
- But it does not change the marks of the smallest *e* and *f*.

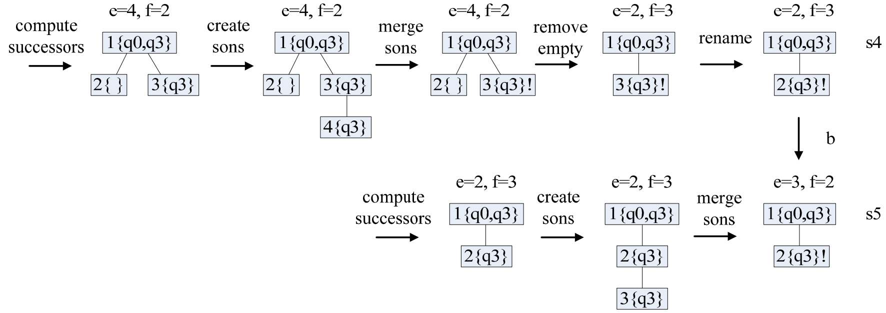


An Example of Construction



An Example of Construction (cont.)





The Determinization

- Let $A = (\Sigma, S, s_0, \delta, F)$ be an NBW with n states.
- An equivalent DPW $D = (\Sigma, S', s_0', \delta', Acc')$:
  - S': the set of compact Safra trees,
  - s_0': an initial compact Safra tree with only one node numbered 1, which is labeled $\{s_0\}$ and has e=2 and f=1,
  - δ': a transition function which, given an input $a \in \Sigma$, transforms a compact Safra tree as described next,
  - The acceptance condition $Acc' = \{F_0, F_1, \ldots, F_{4n}\}$:
    - $F_0 = \{s \in S' \mid f = 1\}$.
    - $F_{2i+1} = \{s \in S' \mid e = i+2 \text{ and } f \geqq e\}$.
    - $F_{2i+2} = \{s \in S' \mid f = i+2 \text{ and } e > f\}$.
    - $i = \{0, 1, 2, \ldots, 2n-1\}$.

The Determinization (cont.)

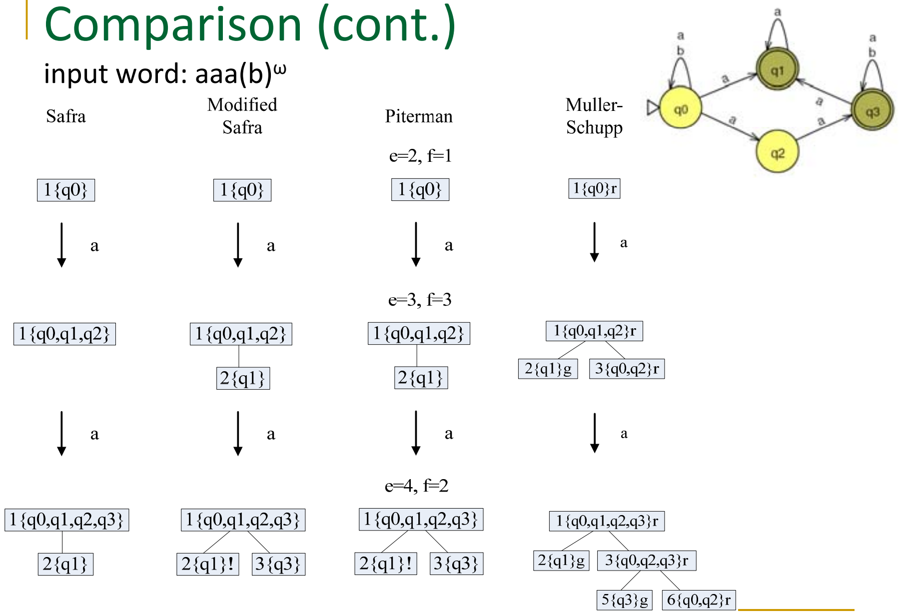
- Steps to compute the next compact Safra-tree state:
  - Replace the label of every node with $\bigcup_{s \in L} \delta(s, a)$.
  - Create a new child with label $L \cap F$.
  - Merge the same states into the leftmost one for each level in the tree.
  - For every node, whose label equals the union of the labels of its children, remove its children and assign the smallest number of these nodes to f.
  - Remove every node with an empty label and set e to the smallest number of removed node.

Comparison

- We define a *modified Safra's construction*, which is similar to the original one, except that we exchange the step of computing successors and the step of creating children.
- Let us compare these four algorithms: Safra, modified Safra, Safra-Piterman, Muller-Schupp.

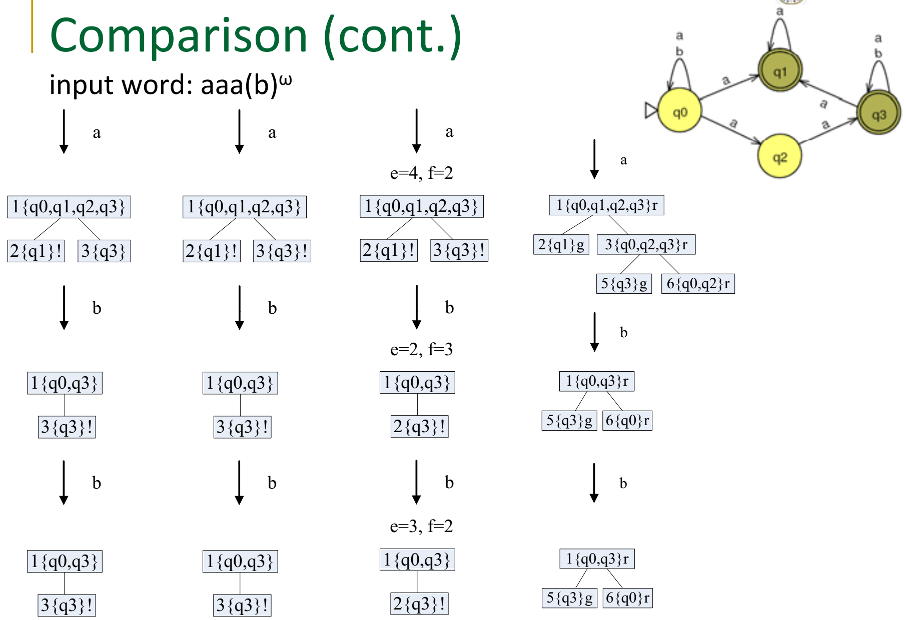
Comparison (cont.)

input word: aaa$(b)^\omega$



Comparison (cont.)

input word: aaa$(b)^\omega$



Some Observations

- Modified Safra trees are slightly better than Safra trees, because a modified Safra tree is usually one step ahead of the corresponding Safra tree.
- Safra-Piterman trees are usually better than modified Safra trees, because a Safra-Piterman tree only cares about the smallest marked name in the tree.
- Modified Safra trees are sometimes better than Safra-Piterman trees, because the rename step spends some time and adds some states.

Some Observations (cont.)

- Muller-Schupp trees are the largest, because they contain more redundant data.
- Safra-Piterman construction performs better than others, because DPW can be translated into NBW more efficiently.
- Muller-Schupp construction helps to understand other algorithms.

Other Complementation Algorithms

- [Thomas]
  - NBW → APW → (complement) NBW
    APW: alternating parity automaton
- [Kupferman and Vardi]
  - NBW → (complement) UCBW → VWAA → NBW
    UCBW: universal co-Büchi automaton
    VWAA: very weak alternating automaton
- There is also a construction (by Kurshan) for DBW complementation, which is quite efficient.

Concluding Remarks

- Büchi complementation is expensive.
- The automata-theoretic approach to model checking tries to avoid it:
  - The system is modeled as a Büchi automaton A.
  - A desired property is given by a PTL formula f.
  - Let B_f ($B_{\sim f}$) denote a Büchi automaton equivalent to f ($\sim f$).
  - The model checking problem translates into
    $$L(A) \subseteq L(B_f) \text{ or } L(A) \cap L(B_{\sim f}) = \emptyset \text{ or } L(A \times B_{\sim f}) = \emptyset.$$
  - So, with PTL to automata translation, the expensive complementation procedure is avoided.
- The well-used model checker SPIN, for example, adopts the automata-theoretic approach and asks the user to express properties in LTL.

Concluding Remarks (cont'd)

- When the B in A⊆ B is given by an arbitrary Büchi automaton, complementation cannot be avoided.
- However, complementation of B may be done "on demand".
- When the containment does not hold, one might find a counterexample before going through the full procedure of complementation.
- There are algorithms for checking language containment based on this idea.
- This line of research is still ongoing.

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