Büchi Automata and Model Checking

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Outline

Introduction

- The simplest computation model for finite behaviors is the finite state automaton , which accepts finite words.
- The simplest computation model for infinite behaviors is the ω-automaton, which accepts infinite words.
- Both have the same syntactic structure.
- Model checking traditionally deals with non-terminating systems.
- Infinite words conveniently represent the infinite behaviors exhibited by a non-terminating system.
- Büchi automata are the simplest kind of ω-automata.
- They were first proposed and studied by J.R. Büchi in the early 1960's, to devise decision procedures for S1S.

Büchi Automata

- A Büchi automaton (BA) has the same structure as a finite state automaton (FA) and is also given by a 5-tuple $(\Sigma, Q, \Delta, q_0, F)$:
  1. Σ is a finite set of symbols (the *alphabet*),
  2. Q is a finite set of *states*,
  3. $\Delta \subseteq Q \times \Sigma \times Q$ is the *transition relation*,
  4. $q_0 \in Q$ is the *start* state (sometimes we allow multiple start states, indicated by Q_0 or Q^0), and
  5. $F \subseteq Q$ is the set of *accepting* states.
- Let $B = (\Sigma, Q, \Delta, q_0, F)$ be a BA and $w = w_1 w_2 \ldots w_i w_{i+1} \ldots$ be an infinite string (or word) over Σ.
- A *run* of B over w is a sequence of states $r_0, r_1, w_2 \ldots, r_i r_{i+1} \ldots$ such that
  1. $r_0 = q_0$ and
  2. $(r_i, w_{i+1}, r_{i+1}) \in \Delta$ for $i \geq 0$.

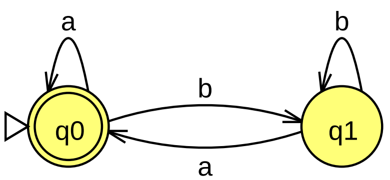
Büchi Automata (cont.)

- Let $inf(\rho)$ denote the set of states occurring infinitely many times in a run ρ.
- An infinite word $w \in \Sigma^\omega$ is *accepted* by a BA B if there exists a run ρ of B over w satisfying the condition:

$$inf(\rho) \cap F \neq \emptyset.$$

- The *language* recognized by B (or the language of B), denoted $L(B)$, is the set of all words that are accepted by B.

An Example Büchi Automaton



- This Büchi automaton accepts infinite words over $\{a, b\}$ that have infinitely many a's.
- Using an ω-regular expression, its language is expressed as $(b^*a)^\omega$.

Closure Properties

- A class of languages is closed under intersection if the intersection of any two languages in the class remains in the class.
- Analogously, for closure under complementation.

Theorem

The class of languages recognizable by Büchi automata is closed under **intersection** *and* **complementation** *(and hence all boolean operations).*

Proof.

Closure under intersection will be proven later by giving a procedure for constructing a Büchi automaton that recognizes the intersection of the languages of two given Büchi automata.
Closure under complementation will be proven in a separate lecture. □

Generalized Büchi Automata

- A generalized Büchi automaton (GBA) has an acceptance component of the form $F = \{F_1, F_2, \cdots, F_n\} \subseteq 2^Q$.
- A run ρ of a GBA is accepting if for each $F_i \in F$, $inf(\rho) \cap F_i \neq \emptyset$.
- GBA's naturally arise in the modeling of finite-state concurrent systems with fairness constraints.
- They are also a convenient intermediate representation in the translation from a linear temporal formula to an equivalent BA.
- There is a simple translation from a GBA to a Büchi automaton, as shown next.

GBA to BA

- Let $B = (\Sigma, Q, \Delta, Q^0, F)$, where $F = \{F_1, \cdots, F_n\}$, be a GBA.
- Construct $B' = (\Sigma, Q \times \{0, \cdots, n\}, \Delta', Q^0 \times \{0\}, Q \times \{n\})$.
- The transition relation Δ' is constructed such that $(\langle q, x\rangle, a, \langle q', y\rangle) \in \Delta'$ when $(q, a, q') \in \Delta$ and x and y are defined according to the following rules:
  - If $q' \in F_i$ and $x = i - 1$, then $y = i$.
  - If $x = n$, then $y = 0$.
  - Otherwise, $y = x$.
- Claim: $L(B') = L(B)$.

Theorem

For every GBA B, there is an equivalent BA B' such that $L(B') = L(B)$.

Model Checking Using Automata (cont.)

- Finite automata can be used to model concurrent and reactive systems as well.
- One of the main advantages of using automata for model checking is that both the modeled system and the specification are represented in the same way.
- A Kripke structure directly corresponds to a Büchi automaton, where all the states are accepting.
- A Kripke structure (S, R, S_0, L) can be transformed into an automaton $A = (\Sigma, S \cup \{\iota\}, \Delta, \{\iota\}, S \cup \{\iota\})$ with $\Sigma = 2^{AP}$ where
  - $(s, \alpha, s') \in \Delta$ for $s, s' \in S$ iff $(s, s') \in R$ and $\alpha = L(s')$ and
  - $(\iota, \alpha, s) \in \Delta$ iff $s \in S_0$ and $\alpha = L(s)$.

Model Checking Using Automata

- Kripke structures are the most commonly used model for concurrent and reactive systems in model checking.
- Let AP be a set of atomic propositions.
- A Kripke structure M over AP is a four-tuple $M = (S, R, S_0, L)$:
  1. S is a finite set of states.
  2. $R \subseteq S \times S$ is a transition relation that must be total, that is, for every state $s \in S$ there is a state $s' \in S$ such that $R(s, s')$.
  3. $S_0 \subseteq S$ is the set of initial states.
  4. $L : S \rightarrow 2^{AP}$ is a function that labels each state with the set of atomic propositions true in that state.

Model Checking Using Automata (cont.)

- The given system is modeled as a Büchi automaton A.
- Suppose the desired property is originally given by a linear temporal formula f.
- Let B_f (resp. $B_{\neg f}$) denote a Büchi automaton equivalent to f (resp. $\neg f$); we will later study how a temporal formula can be translated into an automaton.
- The model checking problem $A \models f$ is equivalent to asking whether

$$L(A) \subseteq L(B_f) \text{ or } L(A) \cap L(B_{\neg f}) = \emptyset.$$

- The well-used model checker SPIN, for example, adopts this automata-theoretic approach.
- So, we are left with two basic problems:
  - Compute the intersection of two Büchi automata.
  - Test the emptiness of the resulting automaton.

Intersection of Büchi Automata

- Let $B_1 = (\Sigma, Q_1, \Delta_1, Q_1^0, F_1)$ and $B_2 = (\Sigma, Q_2, \Delta_2, Q_2^0, F_2)$.
- We can build an automaton for $L(B_1) \cap L(B_2)$ as follows.
- $B_1 \cap B_2 =$ $(\Sigma, Q_1 \times Q_2 \times \{0,1,2\}, \Delta, Q_1^0 \times Q_2^0 \times \{0\}, Q_1 \times Q_2 \times \{2\})$.
- We have $(\langle r, q, x\rangle, a, \langle r', q', y\rangle) \in \Delta$ iff the following conditions hold:
  - $(r, a, r') \in \Delta_1$ and $(q, a, q') \in \Delta_2$.
  - The third component is affected by the accepting conditions of B_1 and B_2.
    - If $x = 0$ and $r' \in F_1$, then $y = 1$.
    - If $x = 1$ and $q' \in F_2$, then $y = 2$.
    - If $x = 2$, then $y = 0$.
    - Otherwise, $y = x$.
- The third component is responsible for guaranteeing that accepting states from both B_1 and B_2 appear infinitely often.

Intersection of Büchi Automata (cont.)

- A simpler intersection may be obtained when all of the states of one of the automata are accepting.
- Assuming all states of B_1 are accepting and that the acceptance set of B_2 is F_2, their intersection can be defined as follows:

$$B_1 \cap B_2 = (\Sigma, Q_1 \times Q_2, \Delta', Q_1^0 \times Q_2^0, Q_1 \times F_2)$$

  where $(\langle r, q\rangle, a, \langle r', q'\rangle) \in \Delta'$ iff $(r, a, r') \in \Delta_1$ and $(q, a, q') \in \Delta_2$.

Checking Emptiness

- Let ρ be an accepting run of a Büchi automaton $B = (\Sigma, Q, \Delta, Q^0, F)$.
- Then, ρ contains infinitely many accepting states from F.
- Since Q is finite, there is some suffix ρ' of ρ such that every state on it appears infinitely many times.
- Each state on ρ' is reachable from any other state on ρ'.
- Hence, the states in ρ' are included in a strongly connected component.
- This component is reachable from an initial state and contains an accepting state.

Checking Emptiness (cont.)

- Conversely, any strongly connected component that is reachable from an initial state and contains an accepting state generates an accepting run of the automaton.
- Thus, checking nonemptiness of $L(B)$ is equivalent to finding a strongly connected component that is reachable from an initial state and contains an accepting state.
- That is, the language $L(B)$ is nonempty iff there is a reachable accepting state with a cycle back to itself.

Double DFS Algorithm

procedure *emptiness*
  for all $q_0 \in Q^0$ **do**
    dfs1(q_0);
  terminate(*True*);
end procedure

procedure *dfs1*(q)
  local q';
  hash(q);
  for all successors q' of q **do**
    if q' not in the hash table **then** *dfs1*(q');
  if *accept*(q) **then** *dfs2*(q);
end procedure

Double DFS Algorithm (cont.)

procedure *dfs2*(q)
  local q';
  flag(q);
  for all successors q' of q **do**
    if q' on *dfs1* stack **then** terminate(*False*);
    else if q' not flagged **then** *dfs2*(q');
    end if;
end procedure

Correctness

Lemma

Let q be a node that does not appear on any cycle. Then the DFS algorithm will backtrack from q only after all the nodes that are reachable from q have been explored and backtracked from.

Theorem

The double DFS algorithm returns a counterexample for the emptiness of the checked automaton B exactly when the language $L(B)$ is not empty.

Correctness (cont.)

- Suppose a second DFS is started from a state q and there is a path from q to some state p on the search stack of the first DFS.
- There are two cases:
  - There exists a path from q to a state on the search stack of the first DFS that contains only unflagged nodes when the second DFS is started from q.
  - On every path from q to a state on the search stack of the first DFS there exists a state r that is already flagged.
- The algorithm will find a cycle in the first case.
- We show that the second case is impossible.

Correctness (cont.)

- Suppose the contrary: On every path from q to a state on the search stack of the first DFS there exists a state r that is already flagged.
- Then there is an accepting state from which a second DFS starts but fails to find a cycle even though one exists.
  - Let q be the first such state.
  - Let r be the first flagged state that is reached from q during the second DFS and is on a cycle through q.
  - Let q' be the accepting state that starts the second DFS in which r was first encountered.
- Thus, according to our assumptions, a second DFS was started from q' before a second DFS was started from q.

Correctness (cont.)

- Case 1: The state q' is reachable from q.
  - There is a cycle $q' \to \cdots \to r \to \cdots \to q \to \cdots \to q'$.
  - This cycle could not have been found previously.
  - This contradicts our assumption that q is the first accepting state from which the second DFS missed a cycle.
- Case 2: The state q' is not reachable from q.
  - q' cannot appear on a cycle.
  - q is reachable from r and q'.
  - If q' does not occur on a cycle, by Lemma 23 we must have backtracked from q in the first DFS before from q'.
  - This contradicts our assumption about the order of doing the second DFS.

Concluding Remarks

- Büchi automata occupy a very special position in logic and automata theory.
- They have found practical applications in linear temporal logic model checking.
- In another lecture, we will study how a linear temporal logic formula can be translated into an equivalent Büchi automaton.

References

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