# Logic

Solutions to Homework for Lecture II

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These are possible solutions to the homework for the second lecture. Some questions can be answered in more than one way, so if your answer differs from mine that does not mean you are wrong. In fact, my solution might be wrong, in which case you should contact me as soon as possible.

## 1 Natural Deduction for Propositional Logic

1. Give a derivation of $(P \wedge Q \to R) \to (P \to Q \to R)$.

$$
{\scriptstyle(\to\mathrm{I}^u)}\,\dfrac{{\scriptstyle(\to\mathrm{I}^v)}\,\dfrac{{\scriptstyle(\to\mathrm{I}^w)}\,\dfrac{{\scriptstyle(\to\mathrm{E})}\,\dfrac{[u\colon P \wedge Q \to R] \qquad {\scriptstyle(\wedge\mathrm{I})}\,\dfrac{[v\colon P] \qquad [w\colon Q]}{P \wedge Q}}{R}}{Q \to R}}{P \to Q \to R}}{(P \wedge Q \to R) \to (P \to Q \to R)}
$$

2. Give a derivation of $(P \to Q \to R) \to (P \to Q) \to P \to R$.

$$
{\scriptstyle(\to\mathrm{I}^u)}\,\dfrac{{\scriptstyle(\to\mathrm{I}^v)}\,\dfrac{{\scriptstyle(\to\mathrm{I}^w)}\,\dfrac{{\scriptstyle(\to\mathrm{E})}\,\dfrac{{\scriptstyle(\to\mathrm{E})}\,\dfrac{[u\colon P \to Q \to R] \qquad [w\colon P]}{Q \to R} \qquad {\scriptstyle(\to\mathrm{E})}\,\dfrac{[v\colon P \to Q] \qquad [w\colon P]}{Q}}{R}}{P \to R}}{(P \to Q) \to R}}{(P \to Q \to R) \to (P \to Q) \to R}
$$

3. Give a derivation of $P \wedge Q \to \neg(P \to \neg Q)$.

$$
{\scriptstyle(\to\mathrm{I}^u)}\,\dfrac{{\scriptstyle(\to\mathrm{I}^v)}\,\dfrac{{\scriptstyle(\to\mathrm{E})}\,\dfrac{{\scriptstyle(\to\mathrm{E})}\,\dfrac{[v\colon P \to \neg Q] \qquad {\scriptstyle(\wedge\mathrm{E}_l)}\,\dfrac{[u\colon P \wedge Q]}{P}}{\neg Q} \qquad {\scriptstyle(\wedge\mathrm{E}_r)}\,\dfrac{[u\colon P \wedge Q]}{Q}}{\bot}}{\neg(P \to \neg Q)}}{P \wedge Q \to \neg(P \to \neg Q)}
$$

4. Give a derivation of $\neg P \vee Q \to (P \to Q)$.

$$
(\to\mathrm{I}^u)\,\dfrac{(\to\mathrm{I}^{v'})\,\dfrac{(\vee\mathrm{E}^{v,w})\,\dfrac{[u\colon \neg P \vee Q] \qquad (\bot\mathrm{E})\,\dfrac{(\to\mathrm{E})\,\dfrac{[v\colon \neg P] \qquad [v'\colon P]}{\bot}}{Q} \qquad [w\colon Q]}{Q}}{P \to Q}}{\neg P \vee Q}
$$

5. Give a derivation of $\neg\neg\bot \to \bot$.

$$
(\to\mathrm{I}^u)\,\dfrac{(\to\mathrm{E})\,\dfrac{[u\colon \neg\neg\bot] \qquad (\to\mathrm{I}^v)\,\dfrac{[v\colon \bot]}{\neg\bot}}{\bot}}{\neg\neg\bot \to \bot}
$$

6. Show that the rule $(\bot\mathrm{E})$ is *admissible* in system NK, i.e. show that from the premise $\bot$ you can derive any formula $\varphi$ in NK.

   Thus, we can use $(\bot\mathrm{E})$ as if it were a rule of NK.

$$
\begin{array}{c} [u\colon \neg\varphi] \\ (\mathrm{DN})\,\dfrac{(\to\mathrm{I}^u)\,\dfrac{\bot}{\neg\neg\varphi}}{\varphi} \end{array}
$$

## 2 Natural Deduction for First Order Logic

1. Can you give a derivation of $(\forall x.\varphi) \to (\exists x.\varphi)$ for any formula $\varphi$? Would you accept this inference step in a mathematical proof? Why or why not?

   Yes:

$$
(\to\mathrm{I}^u)\,\dfrac{(\exists\mathrm{I})\,\dfrac{(\forall\mathrm{E})\,\dfrac{[u\colon \forall x.\varphi]}{\varphi}}{\exists x.\varphi}}{(\forall x.\varphi) \to (\exists x.\varphi)}
$$

   In a mathematical proof, this inference step might seem suspiciuos: on an empty domain, any formula $\forall x.\varphi$ is vacuously held to be true, but any $\exists x.\varphi$ would be false.

   However, in logic we usually assume that there is at least one element in the domain (see Lecture I).

2. Show that $(\forall x.\varphi) \wedge (\forall x.\psi) \vdash_{\mathrm{NJ}} \forall x.\varphi \wedge \psi$ for any formulas $\varphi$ and $\psi$.

$$
{\scriptstyle(\forall \mathrm{I})}\ \dfrac{{\scriptstyle(\wedge \mathrm{I})}\ \dfrac{{\scriptstyle(\forall \mathrm{E})}\ \dfrac{{\scriptstyle(\wedge \mathrm{E}_l)}\ \dfrac{u\colon (\forall x.\varphi)\wedge(\forall x.\psi)}{\forall x.\varphi}}{\varphi} \qquad {\scriptstyle(\forall \mathrm{E})}\ \dfrac{{\scriptstyle(\wedge \mathrm{E}_r)}\ \dfrac{u\colon (\forall x.\varphi)\wedge(\forall x.\psi)}{\forall x.\psi}}{\psi}}{\varphi \wedge \psi}}{\forall x.\varphi \wedge \psi}
$$

3. Show that $\vdash_{\mathrm{NJ}} (\forall x.\varphi) \to \neg(\exists x.\neg\varphi)$.

$$
{\scriptstyle(\to \mathrm{I}^u)}\ \dfrac{{\scriptstyle(\to \mathrm{I}^v)}\ \dfrac{{\scriptstyle(\exists \mathrm{E}^w)}\ \dfrac{[v\colon \exists x.\neg\varphi] \qquad {\scriptstyle(\to \mathrm{E})}\ \dfrac{[w\colon \neg\varphi] \qquad {\scriptstyle(\forall \mathrm{E})}\ \dfrac{[u\colon \forall x.\varphi]}{\varphi}}{\bot}}{\bot}}{\neg(\exists x.\neg\varphi)}}{(\forall x.\varphi) \to (\exists x.\neg\varphi)}
$$

# 3 Natural Deduction for Second Order Logic

1. Show that $\vdash_{\mathrm{NJ}^2} \bot \leftrightarrow (\forall P.P)$.

$$
{\scriptstyle(\wedge \mathrm{I})}\ \dfrac{{\scriptstyle(\to \mathrm{I}^u)}\ \dfrac{{\scriptstyle(\bot \mathrm{E})}\ \dfrac{[u\colon \bot]}{\forall P.P}}{\bot \to (\forall P.P)} \qquad {\scriptstyle(\to \mathrm{I}^v)}\ \dfrac{{\scriptstyle(\forall \mathrm{E})}\ \dfrac{[v\colon \forall P.P]}{\bot}}{(\forall P.P) \to \bot}}{\bot \leftrightarrow (\forall P.P)}
$$

2. Show that $\vdash_{\mathrm{NJ}^2} \varphi \vee \psi \leftrightarrow (\forall P.(\varphi \to P) \to (\psi \to P) \to P)$.

   Note: we can choose $P$ such that it occurs neither in $\varphi$ nor in $\psi$, i.e. $P \notin \mathrm{PL}(\varphi) \cup \mathrm{PL}(\psi)$. This is necessary to fulfill the side conditions in the applications of $(\forall\mathrm{I})$ below.

   First subproof:

$$
{\scriptstyle(\to \mathrm{I}^u)}\ \dfrac{{\scriptstyle(\vee \mathrm{E}^{v,w})}\ \dfrac{[u\colon \varphi \vee \psi] \qquad {\scriptstyle(\forall \mathrm{I})}\ \dfrac{{\scriptstyle(\to \mathrm{I}^r)}\ \dfrac{{\scriptstyle(\to \mathrm{I}^s)}\ \dfrac{{\scriptstyle(\to \mathrm{E})}\ \dfrac{\begin{matrix}[s\colon \psi \to P]\\ [r\colon \varphi \to P]\end{matrix} \qquad [v\colon \varphi]}{P}}{(\psi \to P) \to P}}{(\varphi \to P) \to (\psi \to P) \to P}}{\forall P.(\varphi \to P) \to (\psi \to P) \to P} \qquad {\scriptstyle(\forall \mathrm{I})}\ \dfrac{{\scriptstyle(\to \mathrm{I}^{r'})}\ \dfrac{{\scriptstyle(\to \mathrm{I}^{s'})}\ \dfrac{{\scriptstyle(\to \mathrm{E})}\ \dfrac{\begin{matrix}[r'\colon \varphi \to P]\\ [s'\colon \psi \to P]\end{matrix} \qquad [w\colon \psi]}{P}}{(\psi \to P) \to P}}{(\varphi \to P) \to (\psi \to P) \to P}}{\forall P.(\varphi \to P) \to (\psi \to P) \to P}}{\forall P.(\varphi \to P) \to (\psi \to P) \to P}}{\varphi \vee \psi \to (\forall P.(\varphi \to P) \to (\psi \to P) \to P)}
$$

Second subproof:

$$
\dfrac{\dfrac{\dfrac{[u\colon \forall P.(\varphi \to P) \to (\psi \to P) \to P]}{(\varphi \to \varphi \vee \psi) \to (\psi \to \varphi \vee \psi) \to \varphi \vee \psi}\,(\forall\mathrm{E}) \qquad \dfrac{\dfrac{[v\colon \varphi]}{\varphi \vee \psi}\,(\vee\mathrm{I}_l)}{\varphi \to \varphi \vee \psi}\,(\to\mathrm{I}^v)}{(\psi \to \varphi \vee \psi) \to \varphi \vee \psi}\,(\to\mathrm{E}) \qquad \dfrac{\dfrac{[w\colon \psi]}{\varphi \vee \psi}\,(\vee\mathrm{I}_r)}{\psi \to \varphi \vee \psi}\,(\to\mathrm{I}^w)}{\dfrac{\varphi \vee \psi}{(\forall P.(\varphi \to P) \to (\psi \to P) \to P) \to \varphi \vee \psi}\,(\to\mathrm{I}^u)}\,(\to\mathrm{E})
$$

From these two subproofs, $(\wedge\mathrm{I})$ gives the desired results.