Yesterday

Remember yesterday?

- classical logic: reasoning about truth of formulas
- propositional logic: atomic sentences, composed by connectives
- validity and satisfiability can be decided by truth tables
- formulas can be normalized: NNF, DNF, canonical DNF
- number of connectives can be reduced: functionally complete set
- first order logic: can reason about individuals
- validity and satisfiability are undecidable, but can be checked by semantic definitions
- formulas can be normalized: PNF (but also NNF, DNF)

Logic
Part II: Intuitionistic Logic and Natural Deduction

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Principles of Intuitionistic Logic

Intuitionistic logic advocates a different understanding of what logic is about.

- mathematics is about solving concrete problems
  - find $x, y, z \in \mathbb{N}$ such that $x^2 + y^2 = z^2$
  - given one root of $ax^2 + bx + c = 0$, find the other
  - assuming that π is rational, prove that e is rational, too
- logic abstracts away from concrete problems
- investigates how complex problems are composed from simpler ones
- how complex problems can be solved given solutions of their constituent problems

Outline

Intuitionistic Propositional Logic

Intuitionistic First Order Logic

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Intuitionistic Propositional Logic

Intuitionistic First Order Logic

Intuitionistic Propositional Logic: The Basics

The language of intuitionistic propositional logic is the same as classical propositional logic, but the meaning of formulas is different

- propositional letters represent abstract problems
- more complex problems are formed by using the connectives
- solutions of abstract problems are called *proofs*
- it does not make sense to speak about a formula having a truth value
- we are only interested in how to prove formulas

The Brouwer-Heyting-Kolmogorov Interpretation

Proofs of complex formulas are given in terms of the proofs of their constituents:

- a proof of $\varphi \wedge \psi$ is a proof of φ together with a proof of ψ
- a proof of $\varphi \vee \psi$ is a proof of φ or a proof of ψ
- a proof of $\varphi \rightarrow \psi$ is a procedure that can be seen to produce a proof of ψ from a proof of φ
- there is no proof of $\perp$

Examples

For three propositional letters a, b, c we can prove

- $a \to a$
  Given a proof u of a, we can produce a proof of a, namely u itself.
- $(a \wedge b) \to a$
  Assume we have a proof v of $a \wedge b$. Then we can extract from it a proof of a, since it must contain both a proof of a and a proof of b.
- $a \to (a \vee b)$
- $a \to (b \to a)$
- $(a \to (b \to c)) \to (a \to b) \to (a \to c)$

Examples

For three propositional letters a, b, c we can prove

- $a \longrightarrow a$
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- $a \longrightarrow (a \vee b)$
- $a \longrightarrow (b \longrightarrow a)$
- $(a \longrightarrow (b \longrightarrow c)) \longrightarrow (a \longrightarrow b) \longrightarrow (a \longrightarrow c)$

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- $a \longrightarrow (a \vee b)$
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- $(a \longrightarrow (b \longrightarrow c)) \longrightarrow (a \longrightarrow b) \longrightarrow (a \longrightarrow c)$

Comparison with Classical Propositional Logic

Comparison of “a formula is true” and “a formula has a proof”:

- in CL, to show that $\varphi \vee \psi$ is true, we can
  1. assume that φ is false
  2. then show that ψ is true

  in the second step, we can use the fact that φ is false
- in IL, to give a proof of $\varphi \vee \psi$, we must
  1. either give a proof of φ (no matter whether ψ has one)
  2. or give a proof of ψ (no matter whether φ has one)

For other connectives, the difference is not so marked.

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For other connectives, the difference is not so marked.

Comparison: Example

- in CL, a is true if $\neg a$ and vice versa
- in IL, if $\neg a$ has a proof then there can be no proof of a and vice versa:
  Assume we have a proof u of $\neg a$. Because $\neg a \equiv a \rightarrow \perp$ this means that u is a procedure that produces a proof of $\perp$ given a proof of a. But there is no proof of $\perp$, hence there can be no proof of a.
  Assume that we have a proof v of a. Then there can be no proof of $\neg a$. For assume that we had a proof w of $\neg a$; then w could produce a proof of $\perp$ from v. But this is impossible.

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Comparison: Further Examples

- $a \vee \neg a$ is true in CL; for assume a is false, then $\neg a$ is true
- $a \vee \neg a$ does not seem provable in IL
- in CL, if $\neg\neg a$ is true then so is a; hence $\neg\neg a \rightarrow a$ is a classical tautology
- in IL, there does not seem to be a way to get a proof of a from a proof of $\neg\neg a$
- in CL, $\perp$ is never true; in IL, $\perp$ never has a proof
- in CL, $\perp \rightarrow \varphi$ is true for any φ
- in IL, $\perp \rightarrow \varphi$ is vacuosly provable for any φ (*ex falso quodlibet*, EFQ)

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Excursus: Why EFQ?

- in many fields of mathematics, there are contradictory propositions from which anything is derivable
- for example, if $1 = 0$ were true, then
  - $2 = 1 + 1 = 0 + 0 = 0$, $3 = 1 + 1 + 1 = 0, \ldots$
  - hence: for all $n \in \mathbb{N}$, $n = 0$
  - but also: for all $r \in \mathbb{R}$, $r = r \cdot 1 = r \cdot 0 = 0$

  Thus, any equality between numbers holds, all functions are equal!
- in intuitonistic logic, $\perp$ abstractly represents such a proposition

Formalization: First Step

- we want to formalize the process of forming a proof, in particular a good way to handle *assumptions* (e.g., naming them)
- a diagrammatic *derivation* set out in tree-shape shows how the proof of a complex formula depends on simpler proofs
- in the course of a derivation, assumptions can temporarily be made and later discharged (see examples involving implication)

Example

Here is an informal proof of $a \wedge b \rightarrow b \wedge a$:

1. Assume we have a proof of $a \wedge b$.
2. This proof contains of a proof of a.
3. It also contains a proof of b.
4. So if we take the proof of b and put it together with the proof of a, we obtain a proof of $b \wedge a$.
5. We have shown how to construct a proof of $a \wedge b$ from a proof of $a \wedge b$. This constitutes a proof of $a \wedge b \rightarrow b \wedge a$.

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The Example in Natural Deduction

$$u : a \wedge b$$

- the derivation is a tree with assumptions at the leaves
- assumptions are labeled (here with "*u*")
- the levels correspond to the steps of the informal proof
- derivation steps may *discharge* assumptions (as in the final step)
- discharged assumptions are enclosed in brackets

The Example in Natural Deduction

$$\frac{u\colon a \wedge b}{a}$$

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The Example in Natural Deduction

$$\dfrac{\dfrac{\dfrac{[u\colon a \wedge b]}{b} \qquad \dfrac{[u\colon a \wedge b]}{a}}{b \wedge a}}{a \wedge b \to b \wedge a}$$

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The Calculus NJ of Natural Deduction (Propositional Part)

- the assumption rule: assumptions can be added to the current node at any time

$$x : \varphi$$

- for the connectives, there are introduction and elimination rules
  - the introduction rules specify how to construct proofs
  - the elimination rules specify how to extract the information contained in a proof

The Rules for Conjunction

Conjunction Introduction:

$$(\wedge\mathsf{I})\ \frac{\varphi \qquad \psi}{\varphi \wedge \psi}$$

Conjunction Elimination:

$$(\wedge\mathsf{E}_l)\ \frac{\varphi \wedge \psi}{\varphi}$$

$$(\wedge\mathsf{E}_r)\ \frac{\varphi \wedge \psi}{\psi}$$

Example

$$
(\wedge\mathrm{I})\,\dfrac{(\wedge\mathrm{I})\,\dfrac{(\wedge\mathrm{E}_l)\,\dfrac{u\colon a \wedge (b \wedge c)}{a} \qquad (\wedge\mathrm{E}_l)\,\dfrac{(\wedge\mathrm{E}_r)\,\dfrac{u\colon a \wedge (b \wedge c)}{b \wedge c}}{b}}{a \wedge b} \qquad (\wedge\mathrm{E}_r)\,\dfrac{(\wedge\mathrm{E}_r)\,\dfrac{u\colon a \wedge (b \wedge c)}{b \wedge c}}{c}}{(a \wedge b) \wedge c}
$$

The Rules for Disjunction

Disjunction Introduction:

$$(\vee I_l)\ \frac{\varphi}{\varphi \vee \psi}$$

$$(\vee I_r)\ \frac{\psi}{\varphi \vee \psi}$$

Disjunction Elimination:

$$(\vee E^{v,w})\ \frac{\varphi \vee \psi \qquad \begin{matrix}[v : \varphi] \\ \vdots \\ \vartheta\end{matrix} \qquad \begin{matrix}[w : \psi] \\ \vdots \\ \vartheta\end{matrix}}{\vartheta}$$

All open assumptions from the left subderivation are also open in the two right subderivations.

Example

$$(\vee E^{v,w})\ \dfrac{u : a \vee b \qquad (\vee I_r)\ \dfrac{[v : a]}{b \vee a} \qquad (\vee I_l)\ \dfrac{[w : b]}{b \vee a}}{b \vee a}$$

In the same manner, we can prove $(a \vee b) \vee c$ from the assumption $a \vee (b \vee c)$.

The Rules for Implication

Implication Introduction:

$$\dfrac{\begin{matrix}[x : \varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi}\ (\to\mathrm{I}^x)$$

Implication Elimination (*modus ponens*, MP):

$$\dfrac{\varphi \to \psi \qquad \varphi}{\psi}\ (\to\mathrm{E})$$

Examples

$$(\rightarrow\mathrm{I}^u)\ \frac{[u : a]}{a \rightarrow a}$$

$$(\rightarrow\mathrm{I}^v)\ \dfrac{(\rightarrow\mathrm{I}^w)\ \dfrac{\begin{array}{c}[w : b]\\ [v : a]\end{array}}{b \rightarrow a}}{a \rightarrow b \rightarrow a}$$

The Rules for Falsity

Falsity Introduction:

there is no introduction rule for falsity

Falsity Elimination (EFQ):

$$(\bot\mathsf{E})\ \frac{\bot}{\varphi}$$

Example:

$$\begin{array}{r}(\bot\mathsf{E})\\(\to\mathsf{I}^u)\end{array}\ \dfrac{\dfrac{[u\colon\bot]}{a}}{\bot\to a}$$

Further Examples

$$
(\to \mathrm{I}^u)\ \dfrac{(\to \mathrm{I}^w)\ \dfrac{(\to \mathrm{I}^v)\ \dfrac{(\to \mathrm{E})\ \dfrac{(\to \mathrm{E})\ \dfrac{[u : p \to q] \qquad [v : p]}{q} \qquad [w : \neg q]}{\bot}}{\neg p}}{\neg q \to \neg p}}{(p \to q) \to (\neg q \to \neg p)}
$$

Further Examples

$$(\to\mathrm{I}^u)\,\dfrac{(\land\mathrm{I})\,\dfrac{(\to\mathrm{I}^v)\,\dfrac{(\to\mathrm{E})\,\dfrac{[u\colon (a\lor b)\to c] \qquad (\lor\mathrm{I}_l)\,\dfrac{[v\colon a]}{a\lor b}}{c}}{a\to c} \qquad (\to\mathrm{I}^w)\,\dfrac{(\to\mathrm{E})\,\dfrac{[u\colon (a\lor b)\to c] \qquad (\lor\mathrm{I}_r)\,\dfrac{[w\colon b]}{a\lor b}}{c}}{b\to c}}{(a\to c)\land(b\to c)}}{((a\lor b)\to c)\to(a\to c)\land(b\to c)}$$

Derivability and Theorems

- a context Γ is a list of assumptions, i.e. $\Gamma \equiv x_1 : \varphi_1, \ldots, x_n : \varphi_n$
- the *range* of Γ, written $|\Gamma|$, is the set of assumption formulas in Γ, i.e. the φ_i
- we write $\Gamma \vdash_{\mathrm{NJ}} \varphi$ to mean that φ can be derived from assumptions Γ using the rules of NJ
  for example, $u : p \to q, v : \neg q \vdash_{\mathrm{NJ}} \neg p$
- if Γ is a finite set of formulas, $\Gamma \vdash_{\mathrm{NJ}} \varphi$ is taken to mean that there is some context Δ with $|\Delta| = \Gamma$ and $\Delta \vdash \varphi$
  for example, $p \to q, \neg q \vdash_{\mathrm{NJ}} \neg p$
- if $\vdash_{\mathrm{NJ}} \varphi$ (i.e., φ is derivable without assumptions), then φ is a *theorem* of NJ

Some Theorems

Theorems:

- $(a \to (b \to c)) \to (b \to (a \to c))$
- $(a \to (b \to c)) \to (a \wedge b \to c)$
- $(a \to a \to b) \wedge a \to b$

Non-Theorems:

- $a \vee \neg a$
- $\neg\neg a \to a$
- $\neg(a \wedge b) \to \neg a \vee \neg b$
- $(\neg b \to \neg a) \to (a \to b)$

Theorems:

- $\neg\neg(a \vee \neg a)$
- $a \to \neg\neg a$
- $\neg a \wedge \neg b \to \neg(a \vee b)$
- $(a \to b) \to (\neg b \to \neg a)$

Properties of NJ(I)

Theorem (Soundness Theorem)

The system NJ *is sound: If* $\vdash_{\mathrm{NJ}} \varphi$ *then* $\models \varphi$*, i.e. all theorems are propositional tautologies.*

Consequences of the Soundness Theorem

Corollary

If $\Gamma \vdash_{\mathrm{NJ}} \varphi$ *then* $\Gamma \models \varphi$.

Corollary

The system NJ *is consistent, i.e. there is a propositional formula* φ *such that we do not have* $\vdash_{\mathrm{NJ}} \varphi$.

Proof: Indeed, take $\bot$. If we could derive $\vdash_{\mathrm{NJ}} \bot$, then by the soundness lemma $\models \bot$. But that is not the case.

Consequences of the Soundness Theorem

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If $\Gamma \vdash_{\mathrm{NJ}} \varphi$ *then* $\Gamma \models \varphi$*.*

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The system NJ *is consistent, i.e. there is a propositional formula* φ *such that we do not have* $\vdash_{\mathrm{NJ}} \varphi$*.*

Proof: Indeed, take $\bot$. If we could derive $\vdash_{\mathrm{NJ}} \bot$, then by the soundness lemma $\models \bot$. But that is not the case.

Properties of NJ(II)

- is natural deduction complete for classical logic, i.e. does $\models \varphi$ imply $\vdash_{\mathrm{NJ}} \varphi$?
- no: there are classical tautologies (e.g., $a \vee \neg a$) without a proof in natural deduction
- but we obtain a complete inference system for classical logic if we accept assumptions of the form

$$\varphi \vee \neg\varphi$$

as axioms

Outline

Intuitionistic Propositional Logic

Intuitionistic First Order Logic

Intuitionistic First Order Logic

- the language of intuitionistic first order logic is the same as with classical logic
- the BHK interpretation can be extended to quantified formulas:
  - a proof of $\forall x.\varphi$ is a procedure that can be seen to produce a proof of φ for every value of x
  - a proof of $\exists x.\varphi$ is a value for x together with a proof of φ for this value
- NJ contains introduction and elimination rules for the quantifiers

Comparison with Classical Propositional Logic

Comparison of "a formula is true" and "a formula has a proof" (ctd.):

- in CL, to show that $\exists x.\varphi$ is true, we can
  1. assume that φ is false for all x
  2. then derive a contradiction from this assumption
- in IL, to give a proof of $\exists x.\varphi$, we must present a concrete value for x (called a *witness*) and a proof that φ holds for this x

The existential quantifier of intuitionistic logic is *constructive*.

Rules for the Universal Quantifier

Universal Introduction:

$$(\forall I)\ \frac{\varphi}{\forall x.\varphi}$$

where x cannot occur free in any open assumption

Universal Elimination:

$$(\forall E)\ \frac{\forall x.\varphi}{[x := t]\varphi}$$

for any term t

Example

For any φ, we can build the following derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{u : \forall x.\forall y.\varphi}{\forall y.\varphi}\ (\forall E)}{\varphi}\ (\forall E)}{\forall x.\varphi}\ (\forall I)}{\forall y.\forall x.\varphi}\ (\forall I)
$$

The following attempt to derive $p(x) \to p(y)$ fails due to the variable condition:

$$
\dfrac{\dfrac{\dfrac{[u : p(x)]}{\forall x.p(x)}\ (\forall I)}{p(y)}\ (\forall E)}{p(x) \to p(y)}\ (\to I)
$$

Rules for the Existential Quantifier

Existential Introduction:

$$(\exists \mathrm{I})\ \frac{[x := t]\varphi}{\exists x.\varphi}$$

for any term t

Existential Elimination:

$$(\exists \mathrm{E}^u)\ \frac{\exists x.\varphi \qquad \begin{array}{c}[u : \varphi] \\ \vdots \\ \psi\end{array}}{\psi}$$

where x cannot occur free in any open assumptions on the right and in ψ

All open assumptions from the left subderivation are also open in the right subderivation.

Example

For any φ, we can build the following derivation:

$$
(\exists\mathrm{E}^v)\ \dfrac{u : \exists x.\exists y.\varphi \qquad (\exists\mathrm{E}^w)\ \dfrac{[v : \exists y.\varphi] \qquad (\exists\mathrm{I})\ \dfrac{(\exists\mathrm{I})\ \dfrac{[w : \varphi]}{\exists x.\varphi}}{\exists y.\exists x.\varphi}}{\exists y.\exists x.\varphi}}{\exists y.\exists x.\varphi}
$$

The following attempt to derive $(\exists x.\varphi) \to (\forall x.\varphi)$ fails due to the variable condition, if $x \in \mathrm{FV}(\varphi)$:

$$
(\forall\mathrm{I})\ \dfrac{(\exists\mathrm{E}^v)\ \dfrac{u : \exists x.\varphi \qquad [v : \varphi]}{\varphi}}{\forall x.\varphi}
$$

Example

For any φ and ψ where $x \notin \mathrm{FV}(\varphi)$, we have

$$\varphi \vee \exists x.\psi \vdash_{\mathrm{NJ}} \exists x.\varphi \vee \psi :$$

$$(\vee \mathsf{E}^{u,v})\,\dfrac{t : \varphi \vee (\exists x.\psi) \qquad (\exists \mathsf{I})\,\dfrac{(\vee \mathsf{I}_l)\,\dfrac{[u : \varphi]}{\varphi \vee \psi}}{\exists x.\varphi \vee \psi} \qquad (\exists \mathsf{E}^{w})\,\dfrac{[v : \exists x.\psi] \qquad (\exists \mathsf{I})\,\dfrac{(\vee \mathsf{I}_r)\,\dfrac{[w : \psi]}{\varphi \vee \psi}}{\exists x.\varphi \vee \psi}}{\exists x.\varphi \vee \psi}}{\exists x.\varphi \vee \psi}$$

Example

The following attempt to derive $\forall x.\exists y.x < y \vdash_{\mathrm{NJ}} \exists y.\forall x.x < y$ fails:

$$\dfrac{(\forall E)\ \dfrac{u : \forall x.\exists y.x < y}{\exists y.x < y} \qquad (\exists I)\ \dfrac{(\forall I)\ \dfrac{[v : x < y]}{\forall x.x < y}}{\exists y.\forall x.x < y}}{\exists y.\forall x.x < y}\ (\exists E^{v})$$

Soundness and Completeness of NJ

Theorem (Soundness Theorem)

NJ *is sound with respect to the classical semantics.*

Theorem (Completeness Theorem)

When extended with axioms of the form $\varphi \vee \neg\varphi$, NJ *is complete with respect to the classical semantics.*