Single Context Modal Types

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The Two Variable Rules

$$\frac{x : A \in \Gamma}{\Delta ; \Gamma \vdash x : A} \qquad \frac{x : A \in \Delta}{\Delta ; \Gamma \vdash \underline{x} : A}$$

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$$\frac{x : A \in \Gamma}{\Delta ; \Gamma \vdash x : A} \qquad \frac{x : A \in \Delta}{\Delta ; \Gamma \vdash \underline{x} : A}$$

- Mark each modal variable

The Two Variable Rules

$$\frac{x : A \in \Gamma}{\Delta ; \Gamma \vdash x : A} \qquad \frac{x : A \in \Delta}{\Delta ; \Gamma \vdash \underline{x} : A}$$

- Mark each modal variable
- This is very annoying

An Alternate Presentation

$$A ::= b \mid A \to B \mid \Box A$$

$$q ::= \mathsf{ok} \mid \mathsf{valid}$$

$$\Theta ::= \cdot \mid \Theta, x : A_q$$

$x : A$	$\Rightarrow$ ordinary variable
$x : A$ valid	$\Rightarrow$ modal variable

$$\Theta \vdash \underbrace{e : A_q}$$

Typing judge parameterized by mode

One Context S4

$$\frac{x : A \text{ ok} \in \Theta}{\Theta \vdash x : A \text{ ok}}$$

$$\frac{\Theta, x : A \text{ ok} \vdash e : B \text{ ok}}{\Theta \vdash \lambda x. e : A \to B \text{ ok}} \qquad \frac{\Theta \vdash e_1 : A \to B \text{ ok} \quad \Theta \vdash e_2 : A \text{ ok}}{\Theta \vdash e_1\, e_2 : B \text{ ok}}$$

$$\frac{\Theta \vdash e : A \text{ valid}}{\Theta \vdash \mathrm{box}(e) : \Box A \text{ ok}} \qquad \frac{\Theta \vdash e_1 : \Box A \text{ ok} \quad \Theta, x : A \text{ valid} \vdash e_2 : B \text{ ok}}{\Theta \vdash \mathrm{let}\ \mathrm{box}(x) = e_1\ \mathrm{in}\ e_2 : B \text{ ok}}$$

$$\frac{\Theta^{\Box} \vdash e : A \text{ ok}}{\Theta \vdash e : A \text{ valid}}$$

$$\left.\begin{aligned} (\cdot)^{\Box} &= \cdot \\ (\Theta, x : A \text{ ok})^{\Box} &= \Theta^{\Box} \\ (\Theta, x : A \text{ valid})^{\Box} &= \Theta^{\Box}, x : A \text{ valid} \end{aligned}\right\} \text{Delete the ok variables}$$

Proving Equivalence

Define $|e|$ as

$$
\begin{aligned}
|x| &= x \\
|\underline{x}| &= x \\
|\mathrm{box}(e)| &= \mathrm{box}(|e|) \\
|\mathrm{let\ box}(x) = e_1 \mathrm{\ in\ } e_2| &= \mathrm{let\ box}(x) = |e_1| \mathrm{\ in\ } |e_2| \\
|\lambda x : A . e| &= \lambda x : A . |e| \\
|e_1\ e_2| &= |e_1|\ |e_2|
\end{aligned}
$$

$\boxed{\text{$|e|$ deletes modal variable annotations}}$

Proving Equivalence

Define $\mathrm{split}(\Theta) = \Delta ; \Gamma$

$$\mathrm{split}(\cdot) = \bullet ; \bullet$$
$$\mathrm{split}(\Theta, x{:}A\ ok) = \mathrm{let}\ (\Delta;\Gamma) = \mathrm{split}(\Theta)\ \mathrm{in}\ (\Delta;\ \Gamma, x{:}A)$$
$$\mathrm{split}(\Theta, x{:}A\ valid) = \mathrm{let}\ (\Delta;\Gamma) = \mathrm{split}(\Theta)\ \mathrm{in}\ (\Delta, x{:}A\ ;\ \Gamma)$$

Properties

If $\mathrm{split}(\Theta) = \Delta ; \Gamma$ then

1. $\mathrm{split}(\Theta^{\square}) = \Delta ; \bullet$
2. if $x{:}A\ ok \in \Theta$ then $x{:}A \in \Gamma$
3. if $x{:}A\ valid \in \Theta$ then $x{:}A \in \Delta$

Proof: By induction on Θ

Embedding

If $\quad \Theta \vdash e : A$ ok

then $\exists e', \Delta, \Gamma.$ s.t $(\Delta;\Gamma) = split(\Theta)$, $|e'| = e$, $\Delta;\Gamma \vdash e' : A$

<u>Proof</u>: By induction on $\Theta \vdash e : A$

Embedding

If $\quad \Theta \vdash e : A \text{ ok}$

then $\exists e', \Delta, \Gamma.$ s.t $(\Delta;\Gamma) = split(\Theta)$, $|e'| = e$, $\Delta;\Gamma \vdash e' : A$

Case $\dfrac{x : A_q \in \Theta}{\Theta \vdash x : A_q}$

let $\Delta;\Gamma = split(\Theta)$

- Subcase $q = ok$:

  Then $x : A \in \Gamma$

  $\Delta;\Gamma \vdash x : A$ by rule

  Choose $e' = x$

- Subcase $q = valid$:

  Then $x : A \in \Delta$

  $\Delta;\Gamma \vdash \underline{x} : A$ by rule

  Choose $e' = \underline{x}$

Embedding

If $\quad \Theta \vdash e : A \text{ ok}$
then let $(\Delta;\Gamma) = \text{split}(\Theta)$ and $\exists e'$ s.t. $|e'| = e$ and $\Delta;\Gamma \vdash e' : A$

Case $\dfrac{\Theta, x{:}A \text{ ok} \vdash e : B}{\Theta \vdash \lambda x{:}A.e : A \to B}$

$\text{split}(\Theta, x{:}A \text{ ok}) = \Delta;\ \Gamma, x{:}A$ By definition of split

1. $\Delta; \Gamma, x{:}A \vdash e'' : B$ By induction
2. $|e''| = e$
3. $\Delta; \Gamma \vdash \lambda x.e'' : A \to B$ By rule
4. Choose $e' = \lambda x.e''$
5. Note $|\lambda x.e''| = \lambda x.\,|e''|$ By definition
6. $|\lambda x.e''| = \lambda x.e$ By 2.

Embedding

If $\Theta \vdash e : A$ ok
then let $(\Delta;\Gamma) = split(\Theta)$ and $\exists e'$ s.t. $|e'| = e$ and $\Delta;\Gamma \vdash e' : A$

Case $\dfrac{\Theta \vdash e_1 : A \to B \text{ ok} \quad \Theta \vdash e_2 : A \text{ ok}}{\Theta \vdash e_1\, e_2 : B}$

Let $split(\Theta) = \Delta;\Gamma$

1. $\Delta;\Gamma \vdash e_1' : A \to B$ By induction
2. $|e_1'| = e_1$
3. $\Delta;\Gamma \vdash e_2' : A$ By induction
4. $|e_2'| = e_2$
5. $\Delta;\Gamma \vdash e_1'\, e_2' : B$ By rule
6. Choose $e' = e_1'\, e_2'$
7. Observe $|e_1'\, e_2'| = |e_1'|\, |e_2'|$ By definition
   $= e_1\, |e_2'|$ By 2.
   $= e_1\, e_2$ By 4.

Embedding

If $\Theta \vdash e : A$ ok

then let $(\Delta;\Gamma) = split(\Theta)$ and $\exists e'$ s.t. $|e'| = e$ and $\Delta;\Gamma \vdash e' : A$

Case

$$\dfrac{\dfrac{\Theta^{\square} \vdash e : A\ ok}{\Theta \vdash e : A\ valid}}{\Theta \vdash box(e) : \square A\ ok}$$

Let $split(\Theta) = \Delta;\Gamma$

1. $split(\Theta^{\square}) = \Delta;\cdot$ — By lemma
2. $\Delta;\cdot \vdash e'' : A$ — By induction
3. $|e''| = e$
4. $\Delta;\Gamma \vdash box(e'') : \square A$ — By rule
5. Choose $e' = box(e'')$
6. $|box(e'')| = box(|e''|)$ — By definition
7. $ = box(e)$ — By 3.

Embedding

If $\quad \Theta \vdash e : A \text{ ok}$
then let $(\Delta;\Gamma) = split(\Theta)$ and $\exists e'$ s.t. $|e'| = e$ and $\Delta;\Gamma \vdash e' : A$

$$\text{Case} \quad \frac{\Theta \vdash e_1 : \Box A \text{ ok} \qquad \Theta, x{:}A \text{ valid} \vdash e_2 : B \text{ ok}}{\Theta \vdash \text{let box}(x) = e_1 \text{ in } e_2 : B \text{ ok}}$$

Let $\quad split(\Theta) = \Delta;\Gamma$

1.	$split(\Theta, x{:}A \text{ valid}) = (\Delta, x{:}A);\Gamma$	By definition
2.	$\Delta;\Gamma \vdash e_1' : \Box A$	By induction
3.	$\lvert e_1' \rvert = e_1$	
4.	$\Delta, x{:}A;\Gamma \vdash e_2' : B$	By induction
5.	$\lvert e_2' \rvert = e_2$	
6.	$\Delta;\Gamma \vdash \text{let box}(x) = e_1' \text{ in } e_2' : B$	By rule
7.	Choose $e' = \text{let box}(x) = e_1' \text{ in } e_2'$	
8.	$\lvert \text{let box}(x) = e_1' \text{ in } e_2' \rvert = \text{let box}(x) = e_1 \text{ in } e_2$	By 3, 5

Multiple Modalities

If there are multiple modalities

- We need a context for each one
- n modalities $\Longrightarrow$ n contexts
- The single context style works better with many modalities

Adding Additional Modalities

The $\Box A$ modality must validate:

$$K : \Box A \times \Box B \longrightarrow \Box(A \times B)$$

$$T : \Box A \longrightarrow A$$

$$4 : \Box A \longrightarrow \Box \Box A$$

Adding Additional Modalities

The $\Box A$ modality must validate:

$$K : \Box A \times \Box B \longrightarrow \Box(A \times B)$$

$$T : \Box A \longrightarrow A$$

$$4 : \Box A \longrightarrow \Box\Box A$$

Other, weaker modalities exist

The K modality

The weakest well-behaved modality satisfies:

$$K : \Box A \times \Box B \longrightarrow \Box(A \times B)$$

An Alternate Presentation

$$A ::= b \mid A \to B \mid \Box A \mid \bullet A$$

$$q ::= \mathrm{ok} \mid \mathrm{valid} \mid \mathrm{later}$$

$$\Theta ::= \cdot \mid \Theta, x : A_q$$

$$\Theta \vdash \underbrace{e : A_q}_{\text{Typing judge parameterized by mode}}$$

One Context S4

$$\frac{x : A\ q \in \Theta \qquad q \in \{ok, valid\}}{\Theta \vdash x : A\ ok}$$

$$\frac{\Theta, x : A\ ok \vdash e : B\ ok}{\Theta \vdash \lambda x. e : A \to B\ ok} \qquad \frac{\Theta \vdash e_1 : A \to B\ ok \qquad \Theta \vdash e_2 : A\ ok}{\Theta \vdash e_1\ e_2 : B\ ok}$$

$$\frac{\Theta \vdash e : A\ valid}{\Theta \vdash box(e) : \Box A\ ok} \qquad \frac{\Theta \vdash e_1 : \Box A\ ok \qquad \Theta, x : A\ valid \vdash e_2 : B\ ok}{\Theta \vdash let\ box(x) = e_1\ in\ e_2 : B\ ok}$$

$$\frac{\Theta^{\Box} \vdash e : A\ ok}{\Theta \vdash e : A\ valid}$$

$$\left.\begin{aligned}
(\cdot)^{\Box} &= \cdot \\
(\Theta, x : A\ ok)^{\Box} &= \Theta^{\Box} \\
(\Theta, x : A\ valid)^{\Box} &= \Theta^{\Box}, x : A\ valid
\end{aligned}\right\}\ \text{Delete the ok variables}$$

$$(\Theta, x : A\ later)^{\Box} = \Theta^{\Box}$$

Adding a K-type modality

$$\frac{\Theta \vdash e : A \text{ later}}{\Theta \vdash \delta(e) : \bullet A \text{ ok}}$$

$$\frac{\Theta \vdash e_1 : \bullet A \text{ ok} \qquad \Theta, x : A \text{ later} \vdash e_2 : B \text{ ok}}{\Theta \vdash \text{let } \delta(x) = e_1 \text{ in } e_2 : B \text{ ok}}$$

$$\frac{\Theta^\bullet \vdash e : A \text{ ok}}{\Theta \vdash e : A \text{ later}}$$

$$\boxed{\begin{aligned} (\cdot)^\bullet &= \cdot \\ (\Theta, x : A \text{ valid})^\bullet &= \Theta^\bullet, x : A \text{ valid} \\ (\Theta, x : A \text{ later})^\bullet &= \Theta^\bullet, x : A \text{ ok} \\ (\Theta, x : A \text{ ok})^\bullet &= \Theta^\bullet \end{aligned}}$$

Weakening

$\boxed{\Theta \sqsubseteq \Theta'}$

$$\frac{}{\cdot \sqsubseteq \cdot} \qquad \frac{\Theta \sqsubseteq \Theta'}{\Theta, x:A_q \sqsubseteq \Theta', x:A_q} \qquad \frac{\Theta \sqsubseteq \Theta'}{\Theta \sqsubseteq \Theta', x:A_q}$$

Properties:

1. If $x:A_q \in \Theta_1$ and $\Theta_1 \sqsubseteq \Theta_2$ then $x:A_q \in \Theta_2$
2. If $\Theta_1 \sqsubseteq \Theta_2$ then $\Theta_1^{\bullet} \sqsubseteq \Theta_2^{\bullet}$
3. If $\Theta_1 \sqsubseteq \Theta_2$ then $\Theta_1^{\square} \sqsubseteq \Theta_2^{\square}$
4. $\Theta \sqsubseteq \Theta$
5. If $\Theta_1 \sqsubseteq \Theta_2$ and $\Theta_2 \sqsubseteq \Theta_3$ then $\Theta_1 \sqsubseteq \Theta_2$

The Weakening Lemma

If $\Theta' \sqsupseteq \Theta$ and $\Theta \vdash e : A_q$ then $\Theta' \vdash e : A_q$

Proof: By induction on the derivation $\Theta \vdash e : A_q$

The Weakening Lemma

If $\Theta_2 \sqsupseteq \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash e : A_q$

Case $\dfrac{\Theta_1 \vdash e : A \text{ later}}{\Theta_1 \vdash \delta(e) : \bullet A \text{ ok}}$

1. $\Theta_2 \sqsupseteq \Theta_1$ — By assumption
2. $\Theta_1 \vdash e : A \text{ later}$ — Subderivation
3. $\Theta_2 \vdash e : A \text{ later}$ — Induction
4. $\Theta_2 \vdash \delta(e) : \bullet A \text{ ok}$ — Rule

The Weakening Lemma

If $\Theta_2 \sqsupseteq \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash e : A_q$

Case $\dfrac{\Theta_1^{\bullet} \vdash e : A \text{ ok}}{\Theta_1 \vdash e : A \text{ later}}$

1. $\Theta_2 \sqsupseteq \Theta_1$ — By assumption
2. $\Theta_2^{\bullet} \sqsupseteq \Theta_1^{\bullet}$ — Properties of $\sqsupseteq$
3. $\Theta_1^{\bullet} \vdash e : A \text{ ok}$ — Subderivation
4. $\Theta_2^{\bullet} \vdash e : A \text{ ok}$ — Induction
5. $\Theta_2 \vdash e : A \text{ later}$ — Rule

The Weakening Lemma

If $\Theta_2 \sqsupseteq \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash e : A_q$

Case $\dfrac{\Theta_1 \vdash e : A \text{ valid}}{\Theta_1 \vdash \mathrm{box}(e) : \Box A \text{ ok}}$

1. $\Theta_2 \sqsupseteq \Theta_1$ By assumption
2. $\Theta_1 \vdash e : A$ valid Subderivation
3. $\Theta_2 \vdash e : A$ valid Induction
4. $\Theta_2 \vdash \delta(e) : \Box A$ now Rule

The Weakening Lemma

If $\Theta_2 \sqsupseteq \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash e : A_q$

Case $\dfrac{\Theta_1^{\square} \vdash e : A \text{ ok}}{\Theta_1 \vdash e : A \text{ valid}}$

1. $\Theta_2 \sqsupseteq \Theta_1$	By assumption
2. $\Theta_2^{\square} \sqsupseteq \Theta_1^{\square}$	Properties of $\sqsupseteq$
3. $\Theta_1^{\square} \vdash e : A \text{ ok}$	Subderivation
4. $\Theta_2^{\square} \vdash e : A \text{ ok}$	Induction
5. $\Theta_2 \vdash e : A \text{ valid}$	Rule

The Weakening Lemma

If $\Theta_2 \sqsupseteq \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash e : A_q$

Case $\dfrac{\Theta_1, x : A\ ok \vdash e : B\ ok}{\Theta_1 \vdash \lambda x : A . e\ : A \to B\ ok}$

1. $\Theta_2 \sqsupseteq \Theta_1$ By assumption
2. $\Theta_2, x : A\ ok \sqsupseteq \Theta_1, x : A\ ok$ By rule
3. $\Theta_1, x : A\ ok \vdash e : B\ ok$ Subderivation
4. $\Theta_2, x : A\ ok \vdash e : B\ ok$ Induction
5. $\Theta_2 \vdash \lambda x : A . e\ : A \to B\ ok$ Rule

The Weakening Lemma

If $\Theta_2 \sqsupseteq \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash e : A_q$

Case $$\frac{\Theta_1 \vdash e_1 : A \to B \quad \Theta_1 \vdash e_2 : A}{\Theta_1 \vdash e_1\ e_2 : B}$$

1. $\Theta_2 \sqsupseteq \Theta_1$ By assumption
2. $\Theta_1 \vdash e_1 : A \to B$ Subderivation
3. $\Theta_2 \vdash e_1 : A \to B$ Induction
4. $\Theta_1 \vdash e_2 : A$ Subderivation
5. $\Theta_2 \vdash e_2 : A$ Induction
6. $\Theta_2 \vdash e_1\ e_2 : B$ Rule

The Weakening Lemma

If $\Theta_2 \sqsupseteq \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash e : A_q$

Case $$\frac{\Theta_1 \vdash e_1 : \Box A \qquad \Theta_1, x{:}A\ \text{valid} \vdash e_2 : B\ \text{ok}}{\Theta_1 \vdash \text{let box}(x) = e_1\ \text{in}\ e_2 : B\ \text{ok}}$$

1. $\Theta_2 \sqsupseteq \Theta_1$ — By assumption
2. $\Theta_1 \vdash e_1 : \Box A$ — Subderivation
3. $\Theta_2 \vdash e_1 : \Box A$ — Induction
4. $\Theta_2, x{:}A\ \text{valid} \sqsupseteq \Theta_1, x{:}A\ \text{valid}$ — Rule
5. $\Theta_1, x{:}A\ \text{valid} \vdash e_2 : B\ \text{now}$ — Subderivation
6. $\Theta_2, x{:}A\ \text{valid} \vdash e_2 : B\ \text{now}$ — Induction
7. $\Theta_2 \vdash \text{let box}(x) = e_1\ \text{in}\ e_2 : B\ \text{now}$ — Rule

The Weakening Lemma

If $\Theta_2 \sqsupseteq \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash e : A_q$

Case $\dfrac{\Theta_1 \vdash e_1 : \bullet A \qquad \Theta_1, x : A\ later \vdash e_2 : B\ ok}{\Theta_1 \vdash let\ \delta(x) = e_1\ in\ e_2 : B\ ok}$

1.	$\Theta_2 \sqsupseteq \Theta_1$	By assumption
2.	$\Theta_1 \vdash e_1 : \bullet A$	Subderivation
3.	$\Theta_2 \vdash e_1 : \bullet A$	Induction
4.	$\Theta_2, x : A\ later \sqsupseteq \Theta_1, x : A\ valid$	Rule
5.	$\Theta_1, x : A\ later \vdash e_2 : B\ now$	Subderivation
6.	$\Theta_2, x : A\ later \vdash e_2 : B\ now$	Induction
7.	$\Theta_2 \vdash let\ \delta(x) = e_1\ in\ e_2 : B\ now$	Rule

Substitution

$$\boxed{\Theta \vdash \sigma : \Theta'}$$

$$\frac{}{\Theta \vdash \cdot : \cdot} \qquad \frac{\Theta \vdash \sigma : \Theta' \quad \Theta \vdash e : A_q}{\Theta \vdash (\sigma, e/x\, q) : (\Theta', x : A_q)}$$

$$\begin{aligned}
(\cdot)^{\square} &= \cdot \\
(\sigma, e/x \text{ valid})^{\square} &= \sigma^{\square}, e/x \text{ valid} \\
(\sigma, e/x \text{ later})^{\square} &= \sigma^{\square} \\
(\sigma, e/x \text{ ok})^{\square} &= \sigma^{\square}
\end{aligned}
\qquad
\begin{aligned}
(\cdot)^{\bullet} &= \cdot \\
(\sigma, e/x \text{ valid})^{\bullet} &= \sigma^{\bullet}, e/x \text{ valid} \\
(\sigma, e/x \text{ later})^{\bullet} &= \sigma^{\bullet}, e/x \text{ ok} \\
(\sigma, e/x \text{ ok})^{\bullet} &= \sigma^{\bullet}
\end{aligned}$$

Properties: If $\Theta_2 \vdash \sigma : \Theta_1$ then

1. $\Theta_2^{\square} \vdash \sigma^{\square} : \Theta_1^{\square}$
2. $\Theta_2^{\bullet} \vdash \sigma^{\bullet} : \Theta_1^{\bullet}$

Applying Substitution

$$
\begin{aligned}
[\sigma]\, x &= e && \text{if } (e/x\ q) \in \sigma \ \wedge q \neq \text{later} \\
[\sigma]\, \text{box}(e) &= \text{box}([\sigma^{\square}]\, e) \\
[\sigma]\, \text{let box}(x) = e_1 \text{ in } e_2 &= \text{let box}(x) = [\sigma]\, e_1 \text{ in } [\sigma, x/x]\, e_2 \\
[\sigma]\, \delta(e) &= \delta([\sigma^{\bullet}]\, e) \\
[\sigma]\, \text{let } \delta(x) = e_1 \text{ in } e_2 &= \text{let } \delta(x) = [\sigma]\, e_1 \text{ in } [\sigma, x/x]\, e_2 \\
[\sigma]\, \lambda x{:}A.\, e &= \lambda x{:}A.\, [\sigma, x/x]\, e \\
[\sigma]\, (e_1\ e_2) &= [\sigma]\, e_1\ \ [\sigma]\, e_2
\end{aligned}
$$

$$
\begin{aligned}
[\sigma]^{\text{ok}}\, e &= [\sigma]\, e \\
[\sigma]^{\text{later}}\, e &= [\sigma^{\bullet}]\, e \\
[\sigma]^{\text{valid}}\, e &= [\sigma^{\square}]\, e
\end{aligned}
$$

Weakening + Substitution

If $\Theta_2 \sqsupseteq \Theta_1$ and $\Theta_1 \vdash \sigma : \Theta_0$ then $\Theta_2 \vdash \sigma : \Theta_0$

Proof: By induction on $\Theta_1 \vdash \sigma : \Theta_0$

The Substitution Lemma

If $\Theta_2 \vdash \sigma : \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash [\sigma]^q e : A_q$

Proof: By induction on $\Theta_1 \vdash e : A_q$

The Substitution Lemma

If $\Theta_2 \vdash \sigma : \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash [\sigma]^q e : A_q$

Case: $\dfrac{\Theta_1^\bullet \vdash e : A \text{ ok}}{\Theta_1 \vdash e : A \text{ later}}$

1. $\Theta_2 \vdash \sigma : \Theta_1$ Assumption
2. $\Theta_2^\bullet \vdash \sigma^\bullet : \Theta_1^\bullet$ Property of substitution
3. $\Theta_1^\bullet \vdash e : A \text{ ok}$ Subderivation
4. $\Theta_2^\bullet \vdash [\sigma^\bullet]^{ok} e : A \text{ ok}$ Induction
5. $\Theta_2 \vdash [\sigma^\bullet]^{ok} e : A \text{ later}$ Rule
6. $\Theta_2 \vdash [\sigma]^{later} e : A \text{ later}$ Def. of $[\sigma]^q$

The Substitution Lemma

If $\Theta_2 \vdash \sigma : \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash [\sigma]^q e : A_q$

Case: $$\frac{\Theta_1^{\square} \vdash e : A \text{ ok}}{\Theta_1 \vdash e : A \text{ valid}}$$

1. $\Theta_2 \vdash \sigma : \Theta_1$ — Assumption
2. $\Theta_2^{\square} \vdash \sigma^{\square} : \Theta_1^{\square}$ — Property of substitution
3. $\Theta_1^{\square} \vdash e : A \text{ ok}$ — Subderivation
4. $\Theta_2^{\square} \vdash [\sigma^{\square}]^{ok} e : A \text{ ok}$ — Induction
5. $\Theta_2 \vdash [\sigma^{\square}]^{ok} e : A \text{ valid}$ — Rule
6. $\Theta_2 \vdash [\sigma]^{valid} e : A \text{ valid}$ — Def. of $[\sigma]^q$

The Substitution Lemma

If $\Theta_2 \vdash \sigma : \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash [\sigma]^q e : A_q$

Case: $$\frac{\Theta_1 \vdash e : A \text{ valid}}{\Theta_1 \vdash \mathrm{box}(e) : \Box A \text{ ok}}$$

1. $\Theta_2 \vdash \sigma : \Theta_1$ — Assumption
2. $\Theta_1 \vdash e : A \text{ valid}$ — Subderivation
3. $\Theta_2 \vdash [\sigma]^{valid} e : A \text{ valid}$ — Induction
4. $\Theta_2 \vdash \mathrm{box}([\sigma]^{valid} e) : \Box A \text{ ok}$ — Rule
5. $\Theta_2 \vdash [\sigma]^{ok} \mathrm{box}(e) : \Box A \text{ ok}$ — Def of $[\sigma]^q$ and $[\sigma]$

The Substitution Lemma

If $\Theta_2 \vdash \sigma : \Theta_1$ and $\Theta_1 \vdash e : A_q$ then $\Theta_2 \vdash [\sigma]^q e : A_q$

Case: $$\frac{\Theta_1 \vdash e : A \text{ later}}{\Theta_1 \vdash \delta(e) : \bullet A \text{ ok}}$$

1.	$\Theta_2 \vdash \sigma : \Theta_1$	Assumption
2.	$\Theta_1 \vdash e : A$ later	Subderivation
3.	$\Theta_2 \vdash [\sigma]^{later} e : A$ valid	Induction
4.	$\Theta_2 \vdash \delta([\sigma]^{valid} e) : \bullet A$ ok	Rule
5.	$\Theta_2 \vdash [\sigma]^{ok} \delta(e) : \bullet A$ ok	Def of $[\sigma]^q$ and $[\sigma]$