Modal Logic for Programming

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FLOLAC 2024
Taipei, Taiwan

Propositions and Truth Values

17 is prime

$17 > 19$

21 is even

2 is not odd

Propositions and Truth Values

17 is prime = True

$17 > 19$ = False

21 is even = False

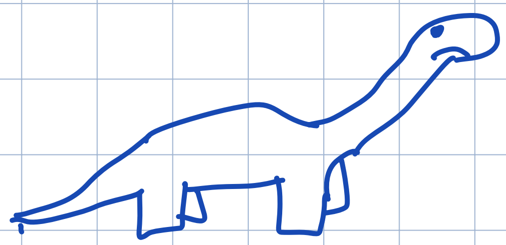
2 is not odd = True

Other Propositions

It is raining.

I am in Taiwan.

Dinosaurs are extinct.



Other Propositions

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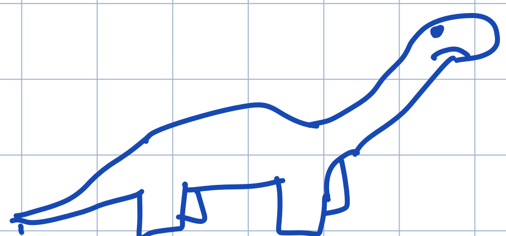
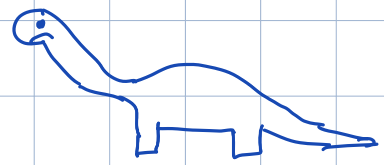


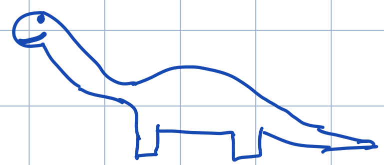
The truth value depends on <u>when</u> the formula is asserted!

Other Propositions

Dinosaurs are extinct.

Time	Truth
Today	True
65 million years ago	False





Other Propositions

Dinosaurs are extinct.

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Today	True
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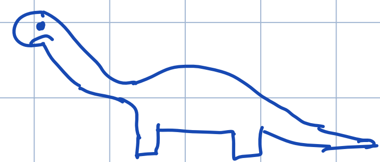

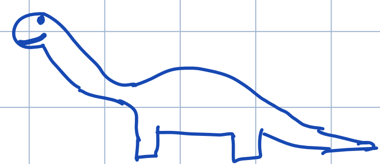

Truth value changes over time

Boolean Logic

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	$\neg P$
T	F
F	T

Boolean Logic

P	Q	$P \wedge Q$
T	T	T
T	F	F
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F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	$\neg P$
T	F
F	T

$\wedge : 2 \times 2 \to 2$

$\vee : 2 \times 2 \to 2$

$\neg : 2 \to 2$

Boolean Logic

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	$\neg P$
T	F
F	T

$$\wedge : 2 \times 2 \to 2$$

$$\vee : 2 \times 2 \to 2$$

$$\neg : 2 \to 2$$

The set of truth values

$\{T, F\}$

Time-Varying Propositions

$$t \models P$$

Time-Varying Propositions

$$t \models P$$

Time

Proposition

Time-Varying Propositions

$$t \models P$$

Time

Proposition

"At time t, P holds"

Semantics of Temporal Logic

$t \models \top$ iff always

Semantics of Temporal Logic

$t \models \top$ iff always

$t \models P \wedge Q$ iff $t \models P$ and $t \models Q$

Semantics of Temporal Logic

$t \models \top$ iff always

$t \models P \wedge Q$ iff $t \models P$ and $t \models Q$

$t \models \bot$ iff never

Semantics of Temporal Logic

$t \models \top$	iff	always
$t \models P \wedge Q$	iff	$t \models P$ and $t \models Q$
$t \models \bot$	iff	never
$t \models P \vee Q$	iff	$t \models P$ or $t \models Q$

Semantics of Temporal Logic

$t \models \top$ iff always

$t \models P \wedge Q$ iff $t \models P$ and $t \models Q$

$t \models \bot$ iff never

$t \models P \vee Q$ iff $t \models P$ or $t \models Q$

$t \models P \Rightarrow Q$ iff if $t \models P$ then $t \models Q$

Semantics of Temporal Logic

$t \models \top$ iff always

$t \models P \wedge Q$ iff $t \models P$ and $t \models Q$

$t \models \bot$ iff never

$t \models P \vee Q$ iff $t \models P$ or $t \models Q$

$t \models P \Rightarrow Q$ iff if $t \models P$ then $t \models Q$

t never changes — where is time?

Temporal Connectives

$t \models \Box P$ iff $\forall t' \geq t.\ t' \models P$

"always P"

$t \models \Diamond P$ iff $\exists t' \geq t.\ t' \models P$

"eventually P"

Examples

Let Rains be true when it rains, false otherwise

Examples

Let Rains be true when it rains, false otherwise

$\Diamond$ Rains "Eventually it rains"

Examples

Let Rains be true when it rains, false otherwise

$\Diamond$ Rains "Eventually it rains"

$\Box$ Rains "Always it rains"

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Let Rains be true when it rains, false otherwise

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$\Box \Diamond$ Rains "Always, it eventually rains"

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Examples

Let Rains be true when it rains, false otherwise

$\Diamond$ Rains "Eventually it rains"

$\Box$ Rains "Always it rains"

$\Box\Diamond$ Rains "Always, it eventually rains"

Some formulas are always true

Examples

Let Rains be true when it rains, false otherwise

$\Diamond$ Rains "Eventually it rains"

$\Box$ Rains "Always it rains"

$\Box \Diamond$ Rains "Always, it eventually rains"

Some formulas are always true

Other formulas are not

Validity

P is valid when

$$\forall t.\ t \models P$$

Validity

P is valid when

$$\forall t.\ t \models P$$

P is true at <u>every</u> time

Validity

P is valid when

$$\forall t.\ t \models P$$

P is true at <u>every</u> time

P is a <u>tautology</u> of temporal logic

Axioms for $\Box$

K: $\Box P \wedge \Box Q \to \Box(P \wedge Q)$

T: $\Box P \to P$

4: $\Box P \to \Box\Box P$

Modal Tautologies

T : $\Box p \rightarrow p$ is valid

Modal Tautologies

$T: \quad \Box p \to p$ is valid

WTS $\quad \forall t.\ t \vDash \Box p \to p$

Modal Tautologies

T : $\Box P \to P$ is valid

WTS $\forall t.\ t \vDash \Box P \to P$

$\equiv \forall t.$ if $t \vDash \Box P$ then $t \vDash P$

Modal Tautologies

$T: \quad \Box P \to P$ is valid

WTS $\quad \forall t.\ t \vDash \Box P \to P$

$\equiv \forall t.$ if $t \vDash \Box P$ then $t \vDash P$

$\equiv \forall t.$ if $\forall t' \geq t.\ t' \vDash P$ then $t \vDash P$

Modal Tautologies

$\forall t.$ if $\forall t' \geq t.\ t' \models P$ then $t \models P$

Proof:

Assume t.

Assume $\forall t' \geq t.\ t' \models P$

Since $t \geq t$, $t \models P$

Modal Tautologies

K : $\Box P \wedge \Box Q \longrightarrow \Box(P \wedge Q)$

We want to show: for all t,

$$t \models \Box P \wedge \Box Q \longrightarrow \Box(P \wedge Q)$$

Modal Tautologies

$t \vDash \Box P \wedge \Box Q \longrightarrow \Box(P \wedge Q)$

$\equiv$ if $t \vDash \Box P \wedge \Box Q$ then $t \vDash \Box(P \wedge Q)$

$\equiv$ if $t \vDash \Box P$ and $t \vDash \Box Q$ then $t \vDash \Box(P \wedge Q)$

Modal Tautologies

if $t \models \Box P$ and $t \models \Box Q$ then $t \models \Box(P \wedge Q)$

Proof:

Assume $t \models \Box P$

$t \models \Box Q$

WTS $t \models \Box(P \wedge Q)$

Modal Tautologies

if $t \models \Box P$ and $t \models \Box Q$ then $t \models \Box(P \wedge Q)$

Proof:

Assume $\forall t' \geq t.\ t' \models P$

$t \models \Box Q$

WTS $t \models \Box(P \wedge Q)$

Modal Tautologies

if $t \vDash \Box P$ and $t \vDash \Box Q$ then $t \vDash \Box(P \wedge Q)$

Proof:

Assume $\forall t' \geq t.\ \ t' \vDash P$

$\forall t' \geq t.\ \ t' \vDash Q$

WTS $t \vDash \Box(P \wedge Q)$

Modal Tautologies

if $t \models \Box P$ and $t \models \Box Q$ then $t \models \Box(P \land Q)$

Proof:

Assume $\forall t' \geq t.\ \ t' \models P$

$\forall t' \geq t.\ \ t' \models Q$

WTS $\forall t' \geq t.\ \ t' \models P \land Q$

Modal Tautologies

if $t \models \Box P$ and $t \models \Box Q$ then $t \models \Box(P \land Q)$

Proof:

Assume $\forall t' \geq t.\ \ t' \models P$

$\forall t' \geq t.\ \ t' \models Q$

WTS $\forall t' \geq t.\ \ t' \models P$ and $t' \models Q$

Modal Tautologies

if $t \models \Box P$ and $t \models \Box Q$ then $t \models \Box(P \wedge Q)$

Proof:

Assume $\forall t' \geq t.\ t' \models P$

$\forall t' \geq t.\ t' \models Q$

WTS $\forall t' \geq t.\ t' \models P$ and $t' \models Q$

Assume $t' \geq t.$

Modal Tautologies

if $t \models \Box P$ and $t \models \Box Q$ then $t \models \Box(P \wedge Q)$

Proof:

Assume $\forall t' \geq t.\ t' \models P$

$\forall t' \geq t.\ t' \models Q$

WTS $\forall t' \geq t.\ t' \models P$ and $t' \models Q$

Assume $t' \geq t$.

$t' \models P$

Modal Tautologies

if $t \models \Box P$ and $t \models \Box Q$ then $t \models \Box(P \wedge Q)$

Proof:

Assume $\forall t' \geq t.\ t' \models P$

$\forall t' \geq t.\ t' \models Q$

WTS $\forall t' \geq t.\ t' \models P$ and $t' \models Q$

Assume $t' \geq t$.

$t' \models P$ and $t' \models Q$

Modal Tautologies

4 : $\Box p \rightarrow \Box\Box p$ is valid

Modal Tautologies

4 : $\Box P \to \Box\Box P$ is valid

WTS $\forall t.\ t \models \Box P \to \Box\Box P$

Modal Tautologies

4: $\Box P \rightarrow \Box\Box P$ is valid

WTS $\forall t.\ t \vDash \Box P \rightarrow \Box\Box P$

$\equiv \forall t.$ if $t \vDash \Box P$ then $t \vDash \Box\Box P$

Modal Tautologies

4 : $\Box P \to \Box\Box P$ is valid

WTS $\forall t.\ t \models \Box P \to \Box\Box P$

$\equiv \forall t.$ if $t \models \Box P$ then $t \models \Box\Box P$

$\equiv \forall t.$ if $(\forall t' \geq t.\ t' \models P)$

then $t \models \Box\Box P$

Modal Tautologies

4 : $\Box P \to \Box\Box P$ is valid

WTS $\forall t.\ t \models \Box P \to \Box\Box P$

$\equiv \forall t.$ if $t \models \Box P$ then $t \models \Box\Box P$

$\equiv \forall t.$ if $(\forall t' \geq t.\ t' \models P)$

then $\forall t' \geq t.\ t' \models \Box P$

Modal Tautologies

4 : $\Box P \to \Box\Box P$ is valid

WTS $\forall t.\ \ t \vDash \Box P \to \Box\Box P$

$\equiv \forall t.$ if $t \vDash \Box P$ then $t \vDash \Box\Box P$

$\equiv \forall t.$ if $(\forall t' \geq t.\ t' \vDash P)$

then $\forall t' \geq t.\ \forall t'' \geq t.\ t'' \vDash P$

Modal Tautologies

$\forall t.\ \text{if } (\forall t' \geq t.\ t' \models P)$

$\quad \text{then } \forall t' \geq t.\ \forall t'' \geq t.\ t'' \models P$

Modal Tautologies

$\forall t.$ if $(\forall t' \geq t.\ t' \vDash P)$

then $\forall t' \geq t.\ \forall t'' \geq t.\ t'' \vDash P$

Proof. Assume t

Modal Tautologies

$\forall t.$ if $(\forall t' \geq t.\ t' \vDash P)$

then $\forall t' \geq t.\ \forall t'' \geq t.\ t'' \vDash P$

Proof. Assume t

Assume $\forall t' \geq t.\ t' \vDash P$

Modal Tautologies

$\forall t.$ if $(\forall t' \geq t.\ t' \models P)$

then $\forall t' \geq t.\ \forall t'' \geq t.\ t'' \models P$

Proof. Assume t
Assume $\forall t' \geq t.\ t' \models P$
Assume $t' \geq t$

Modal Tautologies

$\forall t.$ if $(\forall t' \geq t.\ t' \models P)$

then $\forall t' \geq t.\ \forall t'' \geq t.\ t'' \models P$

Proof. Assume t

Assume $\forall t' \geq t.\ t' \models P$

Assume $t' \geq t$

Assume $t'' \geq t'$.

Modal Tautologies

$\forall t.$ if $(\forall t' \geq t.\ t' \models P)$

then $\forall t' \geq t.\ \forall t'' \geq t.\ t'' \models P$

Proof. Assume t
Assume $\forall t' \geq t.\ t' \models P$
Assume $t' \geq t$
Assume $t'' \geq t'$.
By transitivity, $t'' \geq t$

Modal Tautologies

$\forall t.$ if $(\forall t' \geq t.\ t' \models P)$

then $\forall t' \geq t.\ \forall t'' \geq t.\ t'' \models P$

Proof. Assume t

Assume $\forall t' \geq t.\ t' \models P$

Assume $t' \geq t$

Assume $t'' \geq t'$.

By transitivity, $t'' \geq t$

Hence $t'' \models P$

Axioms for $\Diamond$

K' : $\Box P \wedge \Diamond Q \rightarrow \Diamond (P \wedge Q)$

T' : $P \rightarrow \Diamond P$

4' : $\Diamond \Diamond P \rightarrow P$

Modal Tautology T'

T' : $p \to \Diamond p$ is valid

Modal Tautology T'

T' : $p \to \Diamond p$ is valid

WTS : $\forall t.\ t \models p \to \Diamond p$

Modal Tautology T'

T' : $P \to \Diamond P$ is valid

WTS: $\forall t.\ t \vDash P \to \Diamond P$

$\equiv \forall t.$ if $t \vDash P$ then $t \vDash \Diamond P$

Modal Tautology T'

T' : $p \to \Diamond p$ is valid

WTS: $\forall t.\ t \models p \to \Diamond p$

$\equiv \forall t.$ if $t \models p$ then $t \models \Diamond p$

$\equiv \forall t.$ if $t \models p$ then $\exists t' \geq t.\ t' \models p$

Modal Tautology T'

$\forall t.$ if $t \models p$ then $\exists t' \geq t.$ $t' \models p$

Proof.

Modal Tautology T'

$\forall t.$ if $t \vDash P$ then $\exists t' \geq t.\ t' \vDash P$

Proof. Assume t

Modal Tautology T′

$\forall t.$ if $t \vDash P$ then $\exists t' \geq t.\ t' \vDash P$

Proof. Assume t

Assume $t \vDash P$

Modal Tautology T'

$\forall t.$ if $t \vDash P$ then $\exists t' \geq t.\ t' \vDash P$

Proof. Assume t

Assume $t \vDash P$

Choose t for t'

Modal Tautology T'

$\forall t.$ if $t \models P$ then $\exists t' \geq t.\ t' \models P$

Proof. Assume t

Assume $t \models P$

Choose t for t'

We want to show $t \geq t$ and $t \models P$

Modal Tautology T'

$\forall t.$ if $t \models P$ then $\exists t' \geq t.\ t' \models P$

Proof. Assume t

Assume $t \models P$

Choose t for t'

We want to show $t \geq t$ and $t \models P$

Note $t \geq t$ by reflexivity of $\geq$

Modal Tautology T'

$\forall t.$ if $t \vDash P$ then $\exists t' \geq t.\ t' \vDash P$

Proof. Assume t

Assume $t \vDash P$

Choose t for t'

We want to show $t \geq t$ and $t \vDash P$

Note $t \geq t$ by reflexivity of $\geq$

By assumption, $t \vDash P$

Modal Tautology T'

$\forall t.$ if $t \models P$ then $\exists t' \geq t.\ t' \models P$

Proof. Assume t

Assume $t \models P$

Choose t for t'

We want to show $t \geq t$ and $t \models P$

Note $t \geq t$ by reflexivity

By assumption, $t \models P$

Modal Tautology 4'

4' : $\Diamond\Diamond P \to \Diamond P$ is valid

Modal Tautology 4'

4' : $\Diamond\Diamond p \longrightarrow \Diamond p$ is valid

WTS $\forall t .\ t \models \Diamond\Diamond p \to \Diamond p$

Modal Tautology 4'

4' : $\Diamond\Diamond P \longrightarrow \Diamond P$ is valid

WTS $\forall t.\ \ t \models \Diamond\Diamond P \to \Diamond P$

$\equiv \forall t.$ if $t \models \Diamond\Diamond P$ then $t \models \Diamond P$

Modal Tautology 4'

$4'$: $\Diamond \Diamond P \to \Diamond P$ is valid

WTS $\forall t.\ t \models \Diamond \Diamond P \to \Diamond P$

$\equiv \forall t.$ if $t \models \Diamond \Diamond P$ then $t \models \Diamond P$

$\equiv \forall t.$ if $\exists t' \geq t.\ t' \models \Diamond P$ then $t \models \Diamond P$

Modal Tautology 4'

4': $\Diamond\Diamond P \to \Diamond P$ is valid

WTS $\forall t.\ t \models \Diamond\Diamond P \to \Diamond P$

$\equiv \forall t.$ if $t \models \Diamond\Diamond P$ then $t \models \Diamond P$

$\equiv \forall t.$ if $\exists t' \geq t.\ t' \models \Diamond P$ then $t \models \Diamond P$

$\equiv \forall t$ if $\exists t' \geq t.\ \exists t'' \geq t'.\ t'' \models P$ then $t \models \Diamond P$

Modal Tautology 4'

4' : $\Diamond\Diamond P \to \Diamond P$ is valid

WTS $\forall t.\ t \models \Diamond\Diamond P \to \Diamond P$

$\equiv \forall t.$ if $t \models \Diamond\Diamond P$ then $t \models \Diamond P$

$\equiv \forall t.$ if $\exists t' \geq t.\ t' \models \Diamond P$ then $t \models \Diamond P$

$\equiv \forall t$ if $\exists t' \geq t.\ \exists t'' \geq t'.\ t'' \models P$ then $t \models \Diamond P$

$\equiv \forall t.$ if $\exists t' \geq t,\ t'' \geq t'.\ t'' \models P$ then $\exists t''' \geq t.\ t''' \models P$

Modal Tautology 4'

$\forall t \,.\,$ if $\exists t' \geq t,\ t'' \geq t' \,.\, t'' \models P$ then $\exists t''' \geq t \,.\, t''' \models P$

Modal Tautology 4'

$\forall t$. if $\exists t' \geq t$, $t'' \geq t'$. $t'' \models P$ then $\exists t''' \geq t$. $t''' \models P$

Proof Assume t.

Modal Tautology 4'

$\forall t$. if $\exists t' \geq t,\ t'' \geq t'.\ t'' \models P$ then $\exists t''' \geq t.\ t''' \models P$

Proof Assume t.

Suppose $t' \geq t,\ t'' \geq t',\ t'' \models P$

Modal Tautology 4'

$\forall t$. if $\exists t' \geq t,\ t'' \geq t'.\ t'' \models P$ then $\exists t''' \geq t.\ t''' \models P$

Proof Assume t.

Suppose $t' \geq t$, $t'' \geq t'$, $t'' \models P$

WTS $\exists t''' \geq t.\ t''' \models P$

Modal Tautology 4'

$\forall t$. if $\exists t' \geq t,\ t'' \geq t'.\ t'' \vDash P$ then $\exists t''' \geq t.\ t''' \vDash P$

Proof   Assume t.

Suppose $t' \geq t$, $t'' \geq t'$, $t'' \vDash P$

WTS $\exists t''' \geq t.\ t''' \vDash P$

Choose t''' to be t''

Modal Tautology 4'

$\forall t$. if $\exists t' \geq t,\ t'' \geq t'.\ t'' \vDash P$ then $\exists t''' \geq t.\ t''' \vDash P$

Proof Assume t.

Suppose $t' \geq t,\ t'' \geq t',\ t'' \vDash P$

WTS $\exists t''' \geq t.\ t''' \vDash P$

Choose t''' to be t''

WTS $t'' \geq t$ and $t'' \vDash P$

Modal Tautology 4'

$\forall t$. if $\exists t' \geq t, t'' \geq t'. \; t'' \vDash P$ then $\exists t''' \geq t. \; t''' \vDash P$

Proof Assume t.

Suppose $t' \geq t$, $t'' \geq t'$, $t'' \vDash P$

WTS $\exists t''' \geq t. \; t''' \vDash P$

Choose t''' to be t''

WTS $t'' \geq t$ and $t'' \vDash P$

Since $t'' \geq t'$ and $t' \geq t$ $\quad t'' \geq t$ by transitivity

Modal Tautology 4'

$\forall t.$ if $\exists t' \geq t,\ t'' \geq t'.\ t'' \vDash P$ then $\exists t''' \geq t.\ t''' \vDash P$

Proof Assume t.

Suppose $t' \geq t,\ t'' \geq t',\ t'' \vDash P$

WTS $\exists t''' \geq t.\ t''' \vDash P$

Choose t''' to be t''

WTS $t'' \geq t$ and $t'' \vDash P$

Since $t'' \geq t'$ and $t' \geq t$, $t'' \geq t$ by transitivity

$t'' \vDash P$ by assumption

Modal Tautology K'

K': $\Box P \wedge \Diamond Q \longrightarrow \Diamond(P \wedge Q)$

Modal Tautology K'

K': $\Box P \land \Diamond Q \longrightarrow \Diamond(P \land Q)$

WTS: $\forall t.\quad t \models \Box P \land \Diamond Q \to \Diamond(P \land Q)$

Modal Tautology K'

K': $\Box P \wedge \Diamond Q \longrightarrow \Diamond (P \wedge Q)$

WTS: $\forall t.\ \ t \models \Box P \wedge \Diamond Q \to \Diamond (P \wedge Q)$

$\equiv \forall t.$ if $t \models \Box P \wedge \Diamond Q$ then $t \models \Diamond (P \wedge Q)$

Modal Tautology K'

K': $\Box P \wedge \Diamond Q \longrightarrow \Diamond(P \wedge Q)$

WTS: $\forall t.\ \ t \models \Box P \wedge \Diamond Q \rightarrow \Diamond(P \wedge Q)$

$\equiv \forall t.$ if $t \models \Box P \wedge \Diamond Q$ then $t \models \Diamond(P \wedge Q)$

$\equiv \forall t.$ if $t \models \Box P$ and $t \models \Diamond Q$ then $t \models \Diamond(P \wedge Q)$

Modal Tautology K'

K': $\Box P \land \Diamond Q \longrightarrow \Diamond(P \land Q)$

WTS: $\forall t. \quad t \vDash \Box P \land \Diamond Q \to \Diamond(P \land Q)$

$\equiv \forall t.$ if $t \vDash \Box P \land \Diamond Q$ then $t \vDash \Diamond(P \land Q)$

$\equiv \forall t.$ if $t \vDash \Box P$ and $t \vDash \Diamond Q$ then $t \vDash \Diamond(P \land Q)$

$\equiv \forall t.$ if $(\forall t' \geq t.\ t \vDash P)$ and $t \vDash \Diamond Q$ then $t \vDash \Diamond(P \land Q)$

Modal Tautology K'

K': $\Box P \wedge \Diamond Q \longrightarrow \Diamond(P \wedge Q)$

WTS: $\forall t.\quad t \vDash \Box P \wedge \Diamond Q \rightarrow \Diamond(P \wedge Q)$

$\equiv \forall t.$ if $t \vDash \Box P \wedge \Diamond Q$ then $t \vDash \Diamond(P \wedge Q)$

$\equiv \forall t.$ if $t \vDash \Box P$ and $t \vDash \Diamond Q$ then $t \vDash \Diamond(P \wedge Q)$

$\equiv \forall t.$ if $(\forall t' \geq t.\ t \vDash P)$ and $t \vDash \Diamond Q$ then $t \vDash \Diamond(P \wedge Q)$

$\equiv \forall t.$ if $(\forall t' \geq t.\ t \vDash P)$ and $(\exists t'' \geq t.\ t \vDash Q)$ then $t \vDash \Diamond(P \wedge Q)$

Modal Tautology K'

$\forall t.$ if $(\forall t' \geq t.\ t \models P)$ and $(\exists t'' \geq t.\ t \models Q)$ then $t \models \Diamond(P \wedge Q)$

Modal Tautology K'

$\forall t.$ if $(\forall t' \geq t.\ t \models P)$ and $(\exists t'' \geq t.\ t \models Q)$ then $t \models \Diamond(P \wedge Q)$

Proof: Assume t

Modal Tautology K'

$\forall t.$ if $(\forall t' \geq t.\ t' \models P)$ and $(\exists t'' \geq t.\ t'' \models Q)$ then $t \models \Diamond(P \wedge Q)$

Proof: Assume t

Assume $(\forall t' \geq t.\ t' \models P)$, $t'' \geq t$ and $t'' \models Q$

Modal Tautology K'

$\forall t.$ if $(\forall t' \geq t.\ t' \models P)$ and $(\exists t'' \geq t.\ t'' \models Q)$ then $t \models \Diamond(P \wedge Q)$

Proof: Assume t

Assume $(\forall t' \geq t.\ t' \models P)$, $t'' \geq t$, $t'' \models Q$

WTS $t \models \Diamond(P \wedge Q)$

Modal Tautology K'

$\forall t.$ if $(\forall t' \geq t.\ t' \vDash P)$ and $(\exists t'' \geq t.\ t'' \vDash Q)$ then $t \vDash \Diamond(P \wedge Q)$

Proof: Assume t

Assume $(\forall t' \geq t.\ t' \vDash P)$, $t'' \geq t$, $t'' \vDash Q$

WTS $\exists t' \geq t$ s.t. $t' \vDash P \wedge Q$

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WTS $\exists t' \geq t$ s.t. $t' \vDash P$ and $t' \vDash Q$

Choose t' to be t''

Modal Tautology K'

$\forall t.$ if $(\forall t' \geq t.\ t' \models P)$ and $(\exists t'' \geq t.\ t'' \models Q)$ then $t \models \Diamond(P \wedge Q)$

Proof: Assume t

Assume $(\forall t' \geq t.\ t' \models P)$, $t'' \geq t$, $t'' \models Q$

WTS $\exists t' \geq t$ s.t. $t' \models P$ and $t' \models Q$

Choose t' to be t''

WTS $t'' \geq t$, $t'' \models P$ and $t'' \models Q$

By assumption, $t' \geq t$ and $t'' \models Q$

Modal Tautology K'

$\forall t.$ if $(\forall t' \geq t.\ t \models P)$ and $(\exists t'' \geq t.\ t \models Q)$ then $t \models \Diamond(P \wedge Q)$

Proof: Assume t

Assume $(\forall t' \geq t.\ t' \models P)$, $t'' \geq t$, $t'' \models Q$

WTS $\exists t' \geq t$ s.t. $t' \models P$ and $t' \models Q$

Choose t' to be t''

WTS $t'' \geq t$, $t'' \models P$ and $t'' \models Q$

By assumption, $t' \geq t$ and $t'' \models Q$

Since $t'' \geq t$ and $(\forall t' \geq t.\ t' \models P)$, $t'' \models P$

Summary of Temporal Logic

$$P ::= \top \mid P \wedge Q \mid P \to Q \mid \bot \mid P \vee Q \mid \Box P \mid \Diamond P$$

K: $\Box P \wedge \Box Q \to \Box(P \wedge Q)$

K': $\Box P \wedge \Diamond Q \to \Diamond(P \wedge Q)$

T: $\Box P \to P$

T': $P \to \Diamond P$

4: $\Box P \to \Box\Box P$

4': $\Diamond\Diamond P \to \Diamond P$

Kripke Structures

A **Kripke Frame** is

$$(W \in Set, \quad R \subseteq W \times W)$$

"worlds" "accessibility relation"

Kripke Semantics

A Kripke Model $\_ \models \_ \subseteq W \times Prop$ s.t.

$w \models \bot$ iff never

$w \models P \vee Q$ iff $w \models P$ or $w \models Q$

$w \models \top$ iff always

$w \models P \wedge Q$ iff $w \models P$ and $w \models Q$

$w \models P \to Q$ iff if $w \models P$ then $w \models Q$

$w \models \Box P$ iff $\forall w'.$ if $w R w'$ then $w' \models P$

$w \models \Diamond P$ iff $\exists w'.$ if $w R w'$ then $w' \models P$

S4 Modal Logic

K: $\Box P \wedge \Box Q \rightarrow \Box(P \wedge Q)$

K': $\Box P \wedge \Diamond Q \rightarrow \Diamond(P \wedge Q)$

T: $\Box P \rightarrow P$

T': $P \rightarrow \Diamond P$

4: $\Box P \rightarrow \Box\Box P$

4': $\Diamond\Diamond P \rightarrow \Diamond P$

S4 Modal Logic

K: $\Box P \land \Box Q \rightarrow \Box(P \land Q)$

K': $\Box P \land \Diamond Q \rightarrow \Diamond(P \land Q)$

T: $\Box P \rightarrow P$

T': $P \rightarrow \Diamond P$

4: $\Box P \rightarrow \Box\Box P$

4': $\Diamond\Diamond P \rightarrow \Diamond P$

Frame (W, R)

S4 Modal Logic

K: $\Box P \wedge \Box Q \rightarrow \Box(P \wedge Q)$

K': $\Box P \wedge \Diamond Q \rightarrow \Diamond(P \wedge Q)$

T: $\Box P \rightarrow P$

T': $P \rightarrow \Diamond P$

4: $\Box P \rightarrow \Box\Box P$

4': $\Diamond\Diamond P \rightarrow \Diamond P$

Frame (W, R)

S4 Modal Logic

K: $\Box P \land \Box Q \to \Box (P \land Q)$

K': $\Box P \land \Diamond Q \to \Diamond (P \land Q)$

T: $\Box P \to P$

T': $P \to \Diamond P$

4: $\Box P \to \Box \Box P$

4': $\Diamond \Diamond P \to \Diamond P$

Frame (W, R) is

- Reflexive $\quad wRw$

S4 Modal Logic

K: $\Box P \wedge \Box Q \to \Box(P \wedge Q)$

K': $\Box P \wedge \Diamond Q \to \Diamond(P \wedge Q)$

T: $\Box P \to P$

T': $P \to \Diamond P$

4: $\Box P \to \Box\Box P$

4': $\Diamond\Diamond P \to \Diamond P$

Frame (W, R) is

- Reflexive $\quad wRw$
- Transitive $\quad$ if $w_1 R w_2 \wedge w_2 R w_3$ then $w_1 R w_3$

S4 Modal Logic

K: $\Box P \land \Box Q \to \Box (P \land Q)$

K': $\Box P \land \Diamond Q \to \Diamond (P \land Q)$

T: $\Box P \to P$

T': $P \to \Diamond P$

4: $\Box P \to \Box \Box P$

4': $\Diamond \Diamond P \to \Diamond P$

Frame (W, R) is

- Reflexive $\quad wRw$
- Transitive $\quad$ if $w_1 R w_2 \land w_2 R w_3$ then $w_1 R w_3$
- Such Frames are **S4** modal logic

Kripke Semantics

Different Kripke Semantics

Kripke Semantics

Different Kripke Semantics

Different Modal Logics

Kripke Semantics

Different Kripke Semantics

Different Modal Logics

Different Meanings of $\Box$ and $\Diamond$

Example: Temporal Logic

Kripke Frame $= (\mathrm{Time}, \leq)$

Example: Temporal Logic

Kripke Frame $= (\text{Time}, \leq)$

instants in time

Example: Temporal Logic

Kripke Frame = $(\text{Time}, \leq)$

instants in time

temporal ordering

Example: Temporal Logic

Kripke Frame = (Time, $\leq$)

instants in time

temporal ordering

$\Box P$ = "Always P"

Example: Temporal Logic

Kripke Frame = $(\text{Time}, \leq)$

instants in time

temporal ordering

$\Box P$ = "Always P"

$\Diamond P$ = "Eventually P"

Example 2: Epistemic Logic

Kripke Frame = (Knowledge, $\subseteq$)

Example 2: Epistemic Logic

Kripke Frame = (Knowledge, $\subseteq$)

sets of known facts

Example 2: Epistemic Logic

Kripke Frame = $(\text{Knowledge}, \subseteq)$

sets of known facts

increase of known facts

Example 2: Epistemic Logic

Kripke Frame $= \left(\text{Knowledge}, \quad \subseteq \right)$

sets of known facts

increase of known facts

$\Box P$ = "Always know P"

Example 2: Epistemic Logic

Kripke Frame = (Knowledge, $\subseteq$)

sets of known facts

increase of known facts

$\Box P$ = "Always know P"

$\Diamond P$ = "Possible to know P"

Example 3: Spatial Logic

Kripke Frame $= (L, \rightsquigarrow)$

Example 3: Spatial Logic

Kripke Frame $= (L, \rightsquigarrow)$

locations

Example 3: Spatial Logic

Kripke Frame $= (L, \rightsquigarrow)$

locations

$l \rightsquigarrow l'$ if there is a path between l and l'

Example 3: Spatial Logic

Kripke Frame = $(L, \rightsquigarrow)$

locations

$l \rightsquigarrow l'$ if there is a path between l and l'

$\Box P$ = "P holds at every location reachable from here"

Example 3: Spatial Logic

Kripke Frame $= (L, \rightsquigarrow)$

locations

$l \rightsquigarrow l'$ if there is a path between l and l'

$\Box P$ = "P holds at every location reachable from here"

$\Diamond P$ = "P holds somewhere reachable from here"

Proof Systems for Modal Logic

$\Box P$, $\Diamond P$ work for any Kripke Frame

The current world is implicit

Kripke semantics makes worlds explicit

Natural Deduction

$P, Q ::= \top \mid P \wedge Q \mid P \to Q \mid \bot \mid P \vee Q$

Natural Deduction

$$P, Q ::= \top \mid P \wedge Q \mid P \to Q \mid \bot \mid P \vee Q$$

$$\Gamma ::= \cdot \mid \Gamma, P$$

Natural Deduction

$$P, Q ::= \top \mid P \wedge Q \mid P \to Q \mid \bot \mid P \vee Q$$

$$\Gamma ::= \cdot \mid \Gamma, P$$

$\Gamma \vdash P$ means:

Natural Deduction

$$P, Q ::= \top \mid P \wedge Q \mid P \to Q \mid \bot \mid P \vee Q$$

$$\Gamma ::= \cdot \mid \Gamma, P$$

$\Gamma \vdash P$ means:

"Judge P to be true under assumptions Γ"

Natural Deduction

Natural Deduction

$$\frac{p \in \Gamma}{\Gamma \vdash p}$$

Natural Deduction

$$\frac{p \in \Gamma}{\Gamma \vdash p} \qquad \frac{}{\Gamma \vdash \top}$$

Natural Deduction

$$\frac{P \in \Gamma}{\Gamma \vdash P} \qquad \frac{}{\Gamma \vdash \top}$$

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}$$

Natural Deduction

$$\frac{P \in \Gamma}{\Gamma \vdash P} \qquad \frac{}{\Gamma \vdash \top}$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \qquad \frac{\Gamma \vdash P_1 \wedge P_2}{\Gamma \vdash P_i}$$

Natural Deduction

$$\frac{P \in \Gamma}{\Gamma \vdash P} \qquad \frac{}{\Gamma \vdash \top}$$

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \qquad \frac{\Gamma \vdash P_1 \wedge P_2}{\Gamma \vdash P_i}$$

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \rightarrow Q}$$

Natural Deduction

$$\frac{P \in \Gamma}{\Gamma \vdash P} \qquad \frac{}{\Gamma \vdash \top}$$

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \qquad \frac{\Gamma \vdash P_1 \wedge P_2}{\Gamma \vdash P_i}$$

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q} \qquad \frac{\Gamma \vdash P \to Q \qquad \Gamma \vdash P}{\Gamma \vdash Q}$$

Natural Deduction

Natural Deduction

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash P}$$

Natural Deduction

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash P}$$

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q}$$

Natural Deduction

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash P} \qquad \frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q} \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q}$$

Natural Deduction

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash P} \qquad \frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q} \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q}$$

$$\frac{\Gamma \vdash P \vee Q \qquad \Gamma, P \vdash A \qquad \Gamma, Q \vdash A}{\Gamma \vdash A}$$

Natural Deduction for S4

$P ::= \top \mid P \wedge Q \mid P \to Q \mid \bot \mid P \vee Q \mid \Box P$

Natural Deduction for S4

$$P ::= \top \mid P \wedge Q \mid P \to Q \mid \bot \mid P \vee Q \mid \Box P$$

$$\Gamma ::= \cdot \mid \Gamma, P$$

$$\Delta ::= \cdot \mid \Delta, P$$

Natural Deduction for S4

$$P ::= \top \mid P \wedge Q \mid P \to Q \mid \bot \mid P \vee Q \mid \Box P$$

$$\Gamma ::= \cdot \mid \Gamma, P$$

$$\Delta ::= \cdot \mid \Delta, P$$

$$\Delta ; \Gamma \vdash P$$

"If Γ is true here and Δ is always true, then P is true here"

Natural Deduction for S4

$$\frac{P \in \Gamma}{\Delta ; \Gamma \vdash P} \qquad \frac{}{\Delta ; \Gamma \vdash \top}$$

$$\frac{\Delta ; \Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \qquad \frac{\Delta ; \Gamma \vdash P_1 \wedge P_2}{\Delta ; \Gamma \vdash P_i}$$

$$\frac{\Delta ; \Gamma , P \vdash Q}{\Delta ; \Gamma \vdash P \to Q} \qquad \frac{\Delta ; \Gamma \vdash P \to Q \qquad \Delta ; \Gamma \vdash P}{\Delta ; \Gamma \vdash Q}$$

Natural Deduction for S4

$$\frac{\Delta; \Gamma \vdash \bot}{\Delta; \Gamma \vdash P} \qquad \frac{\Delta; \Gamma \vdash P}{\Delta \Gamma \vdash P \vee Q} \qquad \frac{\Delta; \Gamma \vdash Q}{\Delta; \Gamma \vdash P \vee Q}$$

$$\frac{\Delta; \Gamma \vdash P \vee Q \quad \Delta; \Gamma, P \vdash A \quad \Delta; \Gamma, Q \vdash A}{\Delta; \Gamma \vdash A}$$

Natural Deduction for SH

Natural Deduction for S4

$$\frac{p \in \Delta}{\Delta ; \Gamma \vdash p}$$

Natural Deduction for S4

$$\frac{p \in \Delta}{\Delta ; \Gamma \vdash p} \qquad \frac{\Delta ; \cdot \vdash p}{\Delta ; \Gamma \vdash \Box p}$$

Natural Deduction for S4

$$\frac{P \in \Delta}{\Delta ; \Gamma \vdash P} \qquad \frac{\Delta ; \cdot \vdash P}{\Delta ; \Gamma \vdash \Box P}$$

$$\frac{\Delta ; \Gamma \vdash \Box P \qquad \Delta , P ; \Gamma \vdash Q}{\Delta ; \Gamma \vdash Q}$$

Why These Rules?

$$\Delta ; \Gamma \vdash P$$

"If Γ is true here and Δ is always true, then P is true here"

Think:

$$\Box \Delta \wedge \Gamma \longrightarrow P \quad \text{is valid}$$

The Modal Hypothesis Rule

$$\frac{P_i \in \Delta}{\Delta ; \Gamma \vdash P_i}$$

The Modal Hypothesis Rule

$$\frac{p_i \in \Delta}{\Delta ; \Gamma \vdash p_i}$$

$p_1, \dots p_n$

The Modal Hypothesis Rule

$$\frac{P_i \in \Delta}{\Delta ; \Gamma \vdash P_i}$$

$P_1, \ldots P_n$ $\qquad$ $Q_1, \ldots, Q_k$

The Modal Hypothesis Rule

$$\frac{P_i \in \Delta}{\Delta ; \Gamma \vdash P_i}$$

$P_1, \ldots P_n$ $\qquad$ $Q_1, \ldots, Q_k$

$\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \wedge \ldots Q_k \to P_i$

valid

The Modal Hypothesis Rule

$$\frac{P_i \in \Delta}{\Delta; \Gamma \vdash P_i}$$

$P_1, \ldots P_n \qquad Q_1, \ldots, Q_k$

$\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \wedge \ldots Q_k \to P_i$
valid

$\forall w. \quad w \models \Box P_1 \wedge \ldots \wedge Q_k \to P_i$

The Modal Hypothesis Rule

$$\frac{P_i \in \Delta}{\Delta ; \Gamma \vdash P_i}$$

$P_1, \dots P_n$ $\qquad$ $Q_1, \dots, Q_k$

$\Box P_1 \wedge \dots \wedge \Box P_n \wedge Q_1 \wedge \dots Q_k \rightarrow P_i$ valid

$\forall w.\ w \models \Box P_1 \wedge \dots \wedge Q_k \rightarrow P_i$

$\equiv \forall w.$ if $w \models \Box P_1$ and $\dots$ $w \models Q_k$ then $w \models P_i$

The Modal Hypothesis Rule

$$\frac{P_i \in \Delta}{\Delta ; \Gamma \vdash P_i}$$

$P_1, \ldots P_n$ $\qquad$ $Q_1, \ldots, Q_k$

$\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \wedge \ldots Q_k \rightarrow P_i$ valid

$\forall w.\ \ w \models \Box P_1 \wedge \ldots \wedge Q_k \rightarrow P_i$

$\equiv \forall w.$ if $w \models \Box P_1$ and $\ldots$ $w \models Q_k$ then $w \models P_i$

Proof. Assume $w \models \Box P_1$ and $\ldots$ $w \models \Box P_n$ and $w \models Q_1 \ldots w \models Q_k$.

The Modal Hypothesis Rule

$$\dfrac{P_i \in \Delta}{\Delta ; \Gamma \vdash P_i}$$

$P_1, \ldots P_n$ $\qquad$ $Q_1, \ldots, Q_k$

$\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \wedge \ldots Q_k \rightarrow P_i$ valid

$\forall w.\ w \vDash \Box P_1 \wedge \ldots \wedge Q_k \rightarrow P_i$

$\equiv \forall w.$ if $w \vDash \Box P_1$ and $\ldots$ $w \vDash Q_k$ then $w \vDash P_i$

Proof. Assume $w \vDash \Box P_1$ and $\ldots$ $w \vDash \Box P_n$ and $w \vDash Q_1 \ldots w \vDash Q_k$.

$w \vDash \Box P_i$ by hypothesis

The Modal Hypothesis Rule

$$\frac{P_i \in \Delta}{\Delta ; \Gamma \vdash P_i}$$

$P_1, \ldots P_n$ $\quad$ $Q_1, \ldots, Q_k$

$\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \wedge \ldots Q_k \to P_i$ valid

$\forall w. \quad w \models \Box P_1 \wedge \ldots \wedge Q_k \to P_i$

$\equiv \forall w.$ if $w \models \Box P_1$ and ... $w \models Q_k$ then $w \models P_i$

Proof. Assume $w \models \Box P_1$ and ... $w \models \Box P_n$ and $w \models Q_1 \ldots w \models Q_k$.

$w \models \Box P_i$ by hypothesis

$\forall w'.$ if $w R w'$ then $w' \models P.$

The Modal Hypothesis Rule

$$\frac{P_i \in \Delta}{\Delta ; \Gamma \vdash P_i}$$

$P_1, \ldots P_n$ $\quad$ $Q_1, \ldots, Q_k$

$\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \wedge \ldots Q_k \to P_i$
valid

$\forall w. \quad w \vDash \Box P_1 \wedge \ldots \wedge Q_k \to P_i$

$\equiv \forall w. \;$ if $\; w \vDash \Box P_1$ and $\ldots$ $w \vDash Q_k$ $\;$ then $\; w \vDash P_i$

Proof. Assume $w \vDash \Box P_1$ and $\ldots$ $w \vDash \Box P_n$ and $w \vDash Q_1 \ldots w \vDash Q_k$.

$w \vDash \Box P_i$ $\quad$ by hypothesis

$\forall w'.$ if $w R w'$ then $w' \vDash P_i$

Since $w R w$ by reflexivity of R

The Modal Hypothesis Rule

$$\frac{P_i \in \Delta}{\Delta ; \Gamma \vdash P_i}$$

$P_1, \ldots P_n$ $\qquad$ $Q_1, \ldots, Q_k$

$\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \wedge \ldots Q_k \to P_i$

valid

$\forall w.\ \ w \models \Box P_1 \wedge \ldots \wedge Q_k \to P_i$

$\equiv \forall w.$ if $w \models \Box P_1$ and $\ldots$ $w \models Q_k$ then $w \models P_i$

Proof. Assume $w \models \Box P_1$ and $\ldots$ $w \models \Box P_n$ and $w \models Q_1 \ldots w \models Q_k$.

$w \models \Box P_i$ by hypothesis

$\forall w'.$ if $w R w'$ then $w' \models P$.

Since $w R w$ by reflexivity of R, $w \models P$

Box Introduction

$$\frac{\Delta ; \cdot \vdash R}{\Delta ; \Gamma \vdash \Box R}$$

Box Introduction

$$\frac{\Delta ; \cdot \vdash R}{\Delta ; \Gamma \vdash \Box R}$$

$P_1, \ldots P_n$ $\quad$ $Q_1, \ldots Q_k$

Box Introduction

$$\frac{\Delta \,;\, \cdot \vdash R}{\underbrace{\Delta}_{P_1, \dots P_n} \,;\, \underbrace{\Gamma}_{Q_1, \dots Q_k} \vdash \Box R}$$

$$\left.\begin{array}{l} \text{If } \Box P_1 \wedge \dots \wedge \Box P_n \to R \text{ valid} \\ \text{then } \left(\Box P_1 \wedge \dots \wedge \Box P_n \wedge Q_1 \wedge \dots Q_k\right) \to \Box R \text{ valid} \end{array}\right.$$

Box Introduction

$$\frac{\Delta ; \cdot \vdash R}{\Delta ; \Gamma \vdash \Box R}$$

$P_1, \ldots P_n$ $\quad$ $Q_1, \ldots Q_k$

Lemma: If $w \vDash \Box P$ and $w R w'$ then $w' \vDash \Box P$

Box Introduction

$$\frac{\Delta ; \cdot \vdash R}{\Delta ; \Gamma \vdash \Box R}$$

Δ: $P_1, \ldots P_n$ $\quad$ Γ: $Q_1, \ldots Q_k$

Lemma: If $w \models \Box P$ and $w R w'$ then $w' \models \Box P$

Proof: Assume $w \models \Box P \equiv \forall w'.$ if $w R w'$ then $w' \models P$

Box Introduction

$$\frac{\Delta ; \cdot \vdash R}{\Delta ; \Gamma \vdash \Box R}$$

$P_1, \dots P_n \quad Q_1, \dots Q_k$

Lemma: If $w \vDash \Box P$ and $w R w'$ then $w' \vDash \Box P$

Proof: Assume $w \vDash \Box P \equiv \forall w'.$ if $w R w'$ then $w' \vDash P$

Assume $w R w'$

Box Introduction

$$\frac{\Delta;\ \cdot \vdash R}{\Delta;\Gamma \vdash \Box R}$$

Δ: $P_1, \ldots P_n$ $\quad$ Γ: $Q_1, \ldots Q_k$

Lemma: If $w \models \Box P$ and wRw' then $w' \models \Box P$

Proof: Assume $w \models \Box P \equiv \forall w'.$ if wRw' then $w' \models P$
Assume wRw'
WTS $w' \models \Box P \equiv \forall w''.$ if $w'Rw''$ then $w'' \models P$

Box Introduction

$$\frac{\Delta ; \cdot \vdash R}{\Delta ; \Gamma \vdash \Box R}$$

Δ: $P_1, \ldots P_n$ $\quad$ Γ: $Q_1, \ldots Q_k$

Lemma: If $w \models \Box P$ and $w R w'$ then $w' \models \Box P$

Proof: Assume $w \models \Box P \equiv \forall w'.$ if $w R w'$ then $w' \models P$
Assume $w R w'$
WTS $w' \models \Box P \equiv \forall w''.$ if $w' R w''$ then $w'' \models P$
Since $w R w'$ and $w' R w''$, by transitivity $w R w''$

Box Introduction

$$\frac{\Delta ; \cdot \vdash R}{\Delta ; \Gamma \vdash \Box R}$$

Δ: $P_1, \ldots P_n$ $\quad$ Γ: $Q_1, \ldots Q_k$

Lemma: If $w \vDash \Box P$ and $w R w'$ then $w' \vDash \Box P$

Proof: Assume $w \vDash \Box P \equiv \forall w'.$ if $w R w'$ then $w' \vDash P$

Assume $w R w'$

WTS $w' \vDash \Box P \equiv \forall w''.$ if $w' R w''$ then $w'' \vDash P$

Since $w R w'$ and $w' R w''$, by transitivity $w R w''$

By assumption, $w'' \vDash P$

Box Introduction

If $\Box P_1 \wedge \dots \wedge \Box P_n \to R$ valid

then $\Box P_1 \wedge \dots \wedge \Box P_n \wedge Q_1 \dots \wedge Q_k \to \Box R$ valid

Proof: Assume $\Box P_1 \wedge \dots \wedge \Box P_n \to R$ valid

Box Introduction

If $\Box P_1 \wedge \ldots \wedge \Box P_n \to R$ valid
then $\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \ldots \wedge Q_k \to \Box R$ valid

Proof: Assume $\forall w.$ if $w \models \Box P_1$ and $\ldots$ $w \models \Box P_n$ then $w \models R$

Box Introduction

If $\Box P_1 \wedge \ldots \wedge \Box P_n \rightarrow R$ valid

then $\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \ldots \wedge Q_k \rightarrow \Box R$ valid

Proof: Assume $\forall w.$ if $w \models \Box P_1$ and $\ldots$ $w \models \Box P_n$ then $w \models R$

WTS $\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \ldots \wedge Q_k \rightarrow \Box R$ valid

Box Introduction

If $\Box P_1 \wedge \ldots \wedge \Box P_n \rightarrow R$ valid
then $\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \ldots \wedge Q_k \rightarrow \Box R$ valid

Proof: Assume $\forall w$. if $w \vDash \Box P_1$ and $\ldots$ $w \vDash \Box P_n$ then $w \vDash R$

WTS $\forall w$ if $w \vDash \Box P_1$ and $\ldots$ $w \vDash \Box P_n$ and $\ldots$ $w \vDash Q_k$ then $w \vDash \Box R$

Box Introduction

If $\Box P_1 \wedge \ldots \wedge \Box P_n \to R$ valid

then $\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \ldots \wedge Q_k \to \Box R$ valid

Proof: Assume $\forall w.$ if $w \models \Box P_1$ and ... $w \models \Box P_n$ then $w \models R$

WTS $\forall w$ if $w \models \Box P_1$ and ... $w \models \Box P_n$ and ... $w \models Q_k$ then $w \models \Box R$

Assume w, $w \models \Box P_1, \ldots, w \models \Box P_n, \ldots$ $w \models Q_k$

Box Introduction

If $\Box P_1 \wedge \ldots \wedge \Box P_n \to R$ valid

then $\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \ldots \wedge Q_k \to \Box R$ valid

Proof: Assume $\forall w.$ if $w \models \Box P_1$ and $\ldots$ $w \models \Box P_n$ then $w \models R$

WTS $\forall w$ if $w \models \Box P_1$ and $\ldots$ $w \models \Box P_n$ and $\ldots$ $w \models Q_k$ then $w \models \Box R$

Assume w, $w \models \Box P_1, \ldots, w \models \Box P_n, \ldots$ $w \models Q_k$

WTS $w \models \Box R$

Box Introduction

If $\Box P_1 \wedge \ldots \wedge \Box P_n \to R$ valid
then $\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \ldots \wedge Q_k \to \Box R$ valid

Proof: Assume $\forall w.$ if $w \models \Box P_1$ and $\ldots$ $w \models \Box P_n$ then $w \models R$

WTS $\forall w$ if $w \models \Box P_1$ and $\ldots$ $w \models \Box P_n$ and $\ldots$ $w \models Q_k$ then $w \models \Box R$

Assume w, $w \models \Box P_1, \ldots, w \models \Box P_n, \ldots$ $w \models Q_k$

WTS $\forall w'.$ if $w R w'$ then $w' \models R$

Box Introduction

If $\Box P_1 \wedge \ldots \wedge \Box P_n \to R$ valid
then $\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \ldots \wedge Q_k \to \Box R$ valid

Proof: Assume $\forall w.$ if $w \models \Box P_1$ and $\ldots$ $w \models \Box P_n$ then $w \models R$

WTS $\forall w$ if $w \models \Box P_1$ and $\ldots$ $w \models \Box P_n$ and $\ldots$ $w \models Q_k$ then $w \models \Box R$

Assume w, $w \models \Box P_1, \ldots, w \models \Box P_n, \ldots$ $w \models Q_k$

WTS $\forall w'.$ if $w R w'$ then $w' \models R$

Assume w' s.t. $w R w'$.

Box Introduction

If $\Box P_1 \wedge \ldots \wedge \Box P_n \to R$ valid

then $\Box P_1 \wedge \ldots \wedge \Box P_n \wedge Q_1 \ldots \wedge Q_k \to \Box R$ valid

Proof: Assume $\forall w.$ if $w \vDash \Box P_1$ and $\ldots$ $w \vDash \Box P_n$ then $w \vDash R$

WTS $\forall w$ if $w \vDash \Box P_1$ and $\ldots$ $w \vDash \Box P_n$ and $\ldots$ $w \vDash Q_k$ then $w \vDash \Box R$

Assume w, $w \vDash \Box P_1, \ldots, w \vDash \Box P_n, \ldots$ $w \vDash Q_k$

WTS $\forall w'.$ if $w R w'$ then $w' \vDash R$

Assume w' s.t. $w R w'$.

By lemma, $w' \vDash \Box P_1 \ldots$ $w' \vDash \Box P_n$

Box Introduction

If $\Box P_1 \wedge \dots \wedge \Box P_n \to R$ valid

then $\Box P_1 \wedge \dots \wedge \Box P_n \wedge Q_1 \dots \wedge Q_k \to \Box R$ valid

Proof: Assume $\forall w.$ if $w \models \Box P_1$ and ... $w \models \Box P_n$ then $w \models R$

WTS $\forall w$ if $w \models \Box P_1$ and ... $w \models \Box P_n$ and ... $w \models Q_k$ then $w \models \Box R$

Assume w, $w \models \Box P_1, \dots, w \models \Box P_n, \dots\ w \models Q_k$

WTS $\forall w'$. if $w R w'$ then $w' \models R$

Assume w' s.t. $w R w'$.

By lemma, $w' \models \Box P_1 \dots\ w' \models \Box P_n$

Using the hypothesis, $w' \models R$

Box Elimination

$$\frac{\Delta ; \Gamma \vdash \Box P \qquad \Delta, P ; \Gamma \vdash Q}{\Delta ; \Gamma \vdash Q}$$

If $\Box\Delta \wedge \Gamma \rightarrow \Box P$ is valid
and $\Box\Delta \wedge \Box P \wedge \Gamma \rightarrow Q$ is valid
then $\Box\Delta \wedge \Gamma \rightarrow Q$

Box Elimination

If $\Box \Delta \wedge \Gamma \rightarrow \Box P$ is valid

and $\Box \Delta \wedge \Box P \wedge \Gamma \rightarrow Q$ is valid

then $\Box \Delta \wedge \Gamma \longrightarrow Q$

Box Elimination

If $\Box\Delta \wedge \Gamma \to \Box P$ is valid
and $\Box\Delta \wedge \Box P \wedge \Gamma \to Q$ is valid
then $\Box\Delta \wedge \Gamma \to Q$

Proof Assume $\Box\Delta \wedge \Gamma \to \Box P$ is valid

Box Elimination

If $\Box\Delta \wedge \Gamma \to \Box P$ is valid
and $\Box\Delta \wedge \Box P \wedge \Gamma \to Q$ is valid
then $\Box\Delta \wedge \Gamma \to Q$

Proof Assume $\Box\Delta \wedge \Gamma \to \Box P$ is valid
Assume $\Box\Delta \wedge \Box P \wedge \Gamma \to Q$ is valid

Box Elimination

If $\Box\Delta \wedge \Gamma \rightarrow \Box P$ is valid

and $\Box\Delta \wedge \Box P \wedge \Gamma \rightarrow Q$ is valid

then $\Box\Delta \wedge \Gamma \rightarrow Q$

Proof Assume $\forall w.\ w \models \Box\Delta \wedge \Gamma \rightarrow \Box P$

Assume $\forall w.\ w \models \Box\Delta \wedge \Box P \wedge \Gamma \rightarrow Q$

Box Elimination

If $\Box\Delta \wedge \Gamma \rightarrow \Box P$ is valid
and $\Box\Delta \wedge \Box P \wedge \Gamma \rightarrow Q$ is valid
then $\Box\Delta \wedge \Gamma \rightarrow Q$

Proof Assume $\forall w.$ if $w \models \Box\Delta$ and $w \models \Gamma$ then $w \models \Box P$

Assume $\forall w.$ if $w \models \Box\Delta$ and $w \models \Box P$ and $w \models \Gamma$ then $w \models Q$

Box Elimination

If $\Box\Delta \wedge \Gamma \rightarrow \Box P$ is valid

and $\Box\Delta \wedge \Box P \wedge \Gamma \rightarrow Q$ is valid

then $\Box\Delta \wedge \Gamma \longrightarrow Q$

Proof Assume $\forall w.$ if $w \models \Box\Delta$ and $w \models \Gamma$ then $w \models \Box P$ (1)

Assume $\forall w.$ if $w \models \Box\Delta$ and $w \models \Box P$ and $w \models \Gamma$ then $w \models Q$ (2)

WTS $\forall w.$ if $w \models \Box\Delta$ and $w \models \Gamma$ then $w \models Q$.

Box Elimination

If $\Box\Delta \land \Gamma \rightarrow \Box P$ is valid

and $\Box\Delta \land \Box P \land \Gamma \rightarrow Q$ is valid

then $\Box\Delta \land \Gamma \rightarrow Q$

Proof Assume $\forall w.$ if $w \vDash \Box\Delta$ and $w \vDash \Gamma$ then $w \vDash \Box P$ (1)

Assume $\forall w.$ if $w \vDash \Box\Delta$ and $w \vDash \Box P$ and $w \vDash \Gamma$ then $w \vDash Q$ (2)

WTS $\forall w.$ if $w \vDash \Box\Delta$ and $w \vDash \Gamma$ then $w \vDash Q$.

Assume $w \vDash \Box\Delta$, $w \vDash \Gamma$

Box Elimination

If $\Box\Delta \wedge \Gamma \longrightarrow \Box P$ is valid
and $\Box\Delta \wedge \Box P \wedge \Gamma \longrightarrow Q$ is valid
then $\Box\Delta \wedge \Gamma \longrightarrow Q$

Proof Assume $\forall w.$ if $w \vDash \Box\Delta$ and $w \vDash \Gamma$ then $w \vDash \Box P$ (1)

Assume $\forall w.$ if $w \vDash \Box\Delta$ and $w \vDash \Box P$ and $w \vDash \Gamma$ then $w \vDash Q$ (2)

WTS $\forall w.$ if $w \vDash \Box\Delta$ and $w \vDash \Gamma$ then $w \vDash Q$.

Assume $w \vDash \Box\Delta$, $w \vDash \Gamma$

By (1), $w \vDash \Box P$

Box Elimination

If $\Box\Delta \wedge \Gamma \rightarrow \Box P$ is valid
and $\Box\Delta \wedge \Box P \wedge \Gamma \rightarrow Q$ is valid
then $\Box\Delta \wedge \Gamma \rightarrow Q$

Proof Assume $\forall w.$ if $w \models \Box\Delta$ and $w \models \Gamma$ then $w \models \Box P$ (1)

Assume $\forall w.$ if $w \models \Box\Delta$ and $w \models \Box P$ and $w \models \Gamma$ then $w \models Q$ (2)

WTS $\forall w.$ if $w \models \Box\Delta$ and $w \models \Gamma$ then $w \models Q$.

Assume $w \models \Box\Delta$, $w \models \Gamma$

By (1), $w \models \Box P$

By (2), $w \models Q$

Natural Deduction for S4

$$P ::= \top \mid P \wedge Q \mid P \to Q \mid \bot \mid P \vee Q \mid \Box P \mid \Diamond P$$

$$\Gamma ::= \cdot \mid \Gamma, P$$

$$\Delta ::= \cdot \mid \Delta, P$$

$$\Delta ; \Gamma \vdash P$$

"If Γ is true here and Δ is always true, then P is true here"

Diamond in S4

Diamond in S4

$$\frac{\Delta ; \Gamma \vdash P}{\Delta ; \Gamma \vdash \Diamond P}$$

Diamond in S4

$$\frac{\Delta ; \Gamma \vdash P}{\Delta ; \Gamma \vdash \Diamond P} \qquad \frac{\Delta ; \Gamma \vdash \Diamond P \qquad \Delta ; P \vdash \Diamond Q}{\Delta ; \Gamma \vdash \Diamond Q}$$

Diamond in S4

$$\frac{\Delta ; \Gamma \vdash p}{\Delta ; \Gamma \vdash \Diamond p}$$

Diamond in S4

$$\frac{\Delta ; \Gamma \vdash p}{\Delta ; \Gamma \vdash \Diamond p}$$

"$p \to \Diamond p$"

Diamond in S4

$$\frac{\Delta ; \Gamma \vdash P}{\Delta ; \Gamma \vdash \Diamond P}$$

$$\frac{\Delta ; \Gamma \vdash \Diamond P \qquad \Delta ; P \vdash \Diamond Q}{\Delta ; \Gamma \vdash \Diamond Q}$$

Diamond in S4

$$\frac{\Delta ; \Gamma \vdash P}{\Delta ; \Gamma \vdash \Diamond P} \qquad \frac{\Delta ; \Gamma \vdash \Diamond P \quad \Delta , P \vdash \Diamond Q}{\Delta ; \Gamma \vdash \Diamond Q}$$

Γ is missing!

Diamond in S4

$$\frac{\Delta ; \Gamma \vdash P}{\Delta ; \Gamma \vdash \Diamond P} \qquad \frac{\Delta ; \Gamma \vdash \Diamond P \qquad \Delta , P \vdash \Diamond Q}{\Delta ; \Gamma \vdash \Diamond Q}$$

Γ is missing!

- Γ is at current world

Diamond in S4

$$\frac{\Delta ; \Gamma \vdash P}{\Delta ; \Gamma \vdash \Diamond P} \qquad \frac{\Delta ; \Gamma \vdash \Diamond P \qquad \Delta , P \vdash \Diamond Q}{\Delta ; \Gamma \vdash \Diamond Q}$$

Γ is missing!

- Γ is at current world
- $\Diamond P$ means P holds at future world

Diamond in S4

$$\frac{\Delta ; \Gamma \vdash P}{\Delta ; \Gamma \vdash \Diamond P} \qquad \frac{\Delta ; \Gamma \vdash \Diamond P \qquad \Delta , P \vdash \Diamond Q}{\Delta ; \Gamma \vdash \Diamond Q}$$

Γ is missing!

- Γ is at current world
- $\Diamond P$ means P holds at future world
- So in that future world, Δ is still true, plus P holds

Term Assignment for S4

$A ::= 1 \mid A \times B \mid A \to B \mid 0 \mid A + B$

$\Gamma ::= \cdot \mid \Gamma, x : A$

$e ::= x \mid \langle e, e \rangle \mid \pi_i(e) \mid \lambda x{:}A.e \mid e\,e \mid \mathrm{abort}(e)$
$\quad \mid \mathrm{in}_i(e) \mid \mathrm{case}\,(e, \overrightarrow{\mathrm{in}_i\, x_i \to e_i})$

$\Gamma \vdash e : A$

Term Assignment for S4

$$A ::= 1 \mid A \times B \mid A \to B \mid 0 \mid A + B \mid \Box A \mid \Diamond A$$

$$\Gamma ::= \cdot \mid \Gamma, x : A \qquad \Delta ::= \cdot \mid \Delta, x : A$$

$$\begin{aligned} e ::= {} & x \mid \langle e, e \rangle \mid \pi_i(e) \mid \lambda x{:}A.e \mid e\,e \mid \mathrm{abort}(e) \\ & \mid \mathrm{in}_i(e) \mid \mathrm{case}\,(e, \overrightarrow{\mathrm{in}_i\, x_i \to e_i}) \mid \underline{x} \\ & \mid \mathrm{box}(e) \mid \mathrm{let}\ \mathrm{box}(x) = e \ \mathrm{in}\ e' \\ & \mid \mathrm{dia}(e) \mid \mathrm{let}\ \mathrm{dia}(x) = e \ \mathrm{in}\ e' \end{aligned}$$

$$\Delta; \Gamma \vdash e : A$$

S4 Term Assignment

$$\frac{x : A \in \Delta}{\Delta ; \Gamma \vdash \underline{x} : A} \qquad \frac{\Delta ; \cdot \vdash e : A}{\Delta ; \Gamma \vdash \mathrm{box}(e) : \Box A}$$

$$\frac{\Delta ; \Gamma \vdash e : \Box A \qquad \Delta, x : A ; \Gamma \vdash e' : C}{\Delta ; \Gamma \vdash \mathrm{let}\ \mathrm{box}(x) = e \ \mathrm{in}\ e' : C}$$

S4 Term Assignment

$$\frac{\Delta ; \Gamma \vdash e : A}{\Delta ; \Gamma \vdash \mathrm{dia}(e) : \Diamond A}$$

$$\frac{\Delta ; \Gamma \vdash e : \Diamond A \qquad \Delta ; x : A \vdash e' : \Diamond B}{\Delta ; \Gamma \vdash \mathrm{let}\ \mathrm{dia}(x) = e\ \mathrm{in}\ e' : \Diamond B}$$

Substitution

Main syntactic property of type theory:

$\boxed{\text{Substitution}}$

More complex in case of modal types

Source of Challenge

$$\frac{\Delta ; \cdot \vdash e : A}{\Delta ; \Gamma \vdash \mathrm{box}(e) : \Box A}$$

Source of Challenge

$$\frac{\Delta ; \cdot \vdash e : A}{\Delta ; \Gamma \vdash \mathrm{box}(e) : \Box A}$$

Source of Challenge

$$\frac{\Delta ; \cdot \vdash e : A}{\Delta ; \Gamma \vdash \mathrm{box}(e) : \Box A}$$

Variables go <u>away</u> in subderivations!

Weakening

$\boxed{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}$ $\Delta' ; \Gamma'$ has more variables

Weakening

$$\boxed{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}$$

$\Delta' ; \Gamma'$ has more variables

$$\frac{}{\cdot ; \cdot \sqsubseteq \cdot ; \cdot}$$

Weakening

$$\boxed{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}$$

$\Delta' ; \Gamma'$ has more variables

$$\frac{}{\cdot ; \cdot \sqsubseteq \cdot ; \cdot}$$

$$\frac{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}{\Delta , x : A ; \Gamma \sqsubseteq \Delta' , x : A ; \Gamma} \qquad \frac{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}{\Delta ; \Gamma , x : A \sqsubseteq \Delta' ; \Gamma' , x : A}$$

Weakening

$$\boxed{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}$$

$\Delta' ; \Gamma'$ has more variables

$$\frac{}{\cdot ; \cdot \sqsubseteq \cdot ; \cdot}$$

$$\frac{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}{\Delta, x : A ; \Gamma \sqsubseteq \Delta', x : A ; \Gamma}$$

$$\frac{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}{\Delta ; \Gamma, x : A \sqsubseteq \Delta' ; \Gamma', x : A}$$

$$\frac{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}{\Delta ; \Gamma \sqsubseteq \Delta', x : A ; \Gamma'}$$

$$\frac{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma', x : A}$$

Properties of Weakening

Reflexivity : $\Delta;\Gamma \sqsubseteq \Delta;\Gamma$

Transitivity : If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ and $\Delta';\Gamma' \sqsubseteq \Delta'';\Gamma''$
then $\Delta;\Gamma \sqsubseteq \Delta'';\Gamma''$

Both properties follow by induction

Forgetting

<u>Lemma:</u> If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ then $\Delta ; \bullet \sqsubseteq \Delta' ; \bullet$

Forgetting

<u>Lemma:</u> If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ then $\Delta ; \cdot \sqsubseteq \Delta' ; \cdot$

Proof: By induction on the derivation of $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$

Forgetting

Lemma: If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ then $\Delta ; \cdot \sqsubseteq \Delta' ; \cdot$

Case $$\frac{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma', x : A}$$

Forgetting

<u>Lemma:</u> If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ then $\Delta ; \cdot \sqsubseteq \Delta' ; \cdot$

Case $\dfrac{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma', x : A}$

$\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ Subderivation

Forgetting

<u>Lemma:</u> If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ then $\Delta;\cdot \sqsubseteq \Delta';\cdot$

Case $\dfrac{\Delta;\Gamma \sqsubseteq \Delta';\Gamma'}{\Delta;\Gamma \sqsubseteq \Delta';\Gamma',x:A}$

$\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ Subderivation

$\Delta;\cdot \sqsubseteq \Delta';\cdot$ Induction Hypothesis

Forgetting

<u>Lemma:</u> If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ then $\Delta;\cdot \sqsubseteq \Delta';\cdot$

Case $\dfrac{\Delta;\Gamma \sqsubseteq \Delta';\Gamma'}{\Delta, x:A;\Gamma \sqsubseteq \Delta, x:A;\Gamma'}$

Forgetting

<u>Lemma:</u> If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ then $\Delta;\cdot \sqsubseteq \Delta';\cdot$

Case $$\frac{\Delta;\Gamma \sqsubseteq \Delta';\Gamma'}{\Delta, x:A;\Gamma \sqsubseteq \Delta', x:A;\Gamma'}$$

$\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ Subderivation

Forgetting

Lemma: If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ then $\Delta ; \cdot \sqsubseteq \Delta' ; \cdot$

Case $$\frac{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}{\Delta , x : A ; \Gamma \sqsubseteq \Delta , x : A ; \Gamma'}$$

$\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ Subderivation

$\Delta ; \cdot \sqsubseteq \Delta' ; \cdot$ Induction

Forgetting

<u>Lemma:</u> If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ then $\Delta ; \cdot \sqsubseteq \Delta' ; \cdot$

Case $\dfrac{\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'}{\Delta, x{:}A ; \Gamma \sqsubseteq \Delta, x{:}A ; \Gamma'}$

$\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ Subderivation

$\Delta ; \cdot \sqsubseteq \Delta' ; \cdot$ Induction

$\Delta, x{:}A ; \cdot \sqsubseteq \Delta', x{:}A ; \cdot$ Rule

Weakening Lemma

If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ and $\Delta ; \Gamma \vdash e : A$ then $\Delta' ; \Gamma' \vdash e : A$

Weakening Lemma

If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ and $\Delta;\Gamma \vdash e : A$ then $\Delta';\Gamma' \vdash e : A$

Proof: induction on the derivation of $\Gamma;\Delta \vdash e : A$

Weakening Lemma

If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ and $\Delta ; \Gamma \vdash e : A$ then $\Delta' ; \Gamma' \vdash e : A$

Case: $\dfrac{\Delta ; \cdot \vdash e : A}{\Delta ; \Gamma \vdash \mathrm{box}(e) : \Box A}$

Weakening Lemma

If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ and $\Delta ; \Gamma \vdash e : A$ then $\Delta' ; \Gamma' \vdash e : A$

Case: $$\frac{\Delta ; \cdot \vdash e : A}{\Delta ; \Gamma \vdash \mathrm{box}(e) : \Box A}$$

$\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ By assumption

Weakening Lemma

If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ and $\Delta;\Gamma \vdash e : A$ then $\Delta';\Gamma' \vdash e : A$

Case: $$\frac{\Delta;\cdot \vdash e : A}{\Delta;\Gamma \vdash \mathrm{box}(e) : \square A}$$

$\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ By assumption

$\Delta;\cdot \vdash e : A$ Subderivation

Weakening Lemma

If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ and $\Delta ; \Gamma \vdash e : A$ then $\Delta' ; \Gamma' \vdash e : A$

Case: $$\frac{\Delta ; \cdot \vdash e : A}{\Delta ; \Gamma \vdash \mathrm{box}(e) : \Box A}$$

$\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ By assumption

$\Delta ; \cdot \vdash e : A$ Subderivation

$\Delta ; \cdot \sqsubseteq \Delta' ; \cdot$ Forgetting Lemma

Weakening Lemma

If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ and $\Delta ; \Gamma \vdash e : A$ then $\Delta' ; \Gamma' \vdash e : A$

Case: $$\frac{\Delta ; \cdot \vdash e : A}{\Delta ; \Gamma \vdash \mathrm{box}(e) : \Box A}$$

$\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$	By assumption
$\Delta ; \cdot \vdash e : A$	Subderivation
$\Delta ; \cdot \sqsubseteq \Delta' ; \cdot$	Forgetting Lemma
$\Delta' ; \cdot \vdash e : A$	Induction

Weakening Lemma

If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ and $\Delta ; \Gamma \vdash e : A$ then $\Delta' ; \Gamma' \vdash e : A$

Case: $$\frac{\Delta ; \cdot \vdash e : A}{\Delta ; \Gamma \vdash \mathrm{box}(e) : \Box A}$$

$\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$	By assumption
$\Delta ; \cdot \vdash e : A$	Subderivation
$\Delta ; \cdot \sqsubseteq \Delta' ; \cdot$	Forgetting Lemma
$\Delta' ; \cdot \vdash e : A$	Induction
$\Delta' ; \Gamma' \vdash \mathrm{box}(e) : \Box A$	Rule

Weakening Lemma

If $\Delta; \Gamma \sqsubseteq \Delta'; \Gamma'$ and $\Delta; \Gamma \vdash e : A$ then $\Delta'; \Gamma' \vdash e : A$

Case $$\frac{\Delta; \Gamma, x : A \vdash e : B}{\Delta; \Gamma \vdash \lambda x. e : A \to B}$$

Weakening Lemma

If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ and $\Delta ; \Gamma \vdash e : A$ then $\Delta' ; \Gamma' \vdash e : A$

Case $\dfrac{\Delta ; \Gamma , x : A \vdash e : B}{\Delta ; \Gamma \vdash \lambda x . e : A \to B}$

$\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ Assumption

Weakening Lemma

If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ and $\Delta ; \Gamma \vdash e : A$ then $\Delta' ; \Gamma' \vdash e : A$

Case $\dfrac{\Delta ; \Gamma , x : A \vdash e : B}{\Delta ; \Gamma \vdash \lambda x . e : A \to B}$

$\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ Assumption

$\Delta ; \Gamma , x : A \vdash e : B$ Subderivation

Weakening Lemma

If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ and $\Delta;\Gamma \vdash e : A$ then $\Delta';\Gamma' \vdash e : A$

Case $$\frac{\Delta;\Gamma, x : A \vdash e : B}{\Delta;\Gamma \vdash \lambda x.e : A \to B}$$

$\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ Assumption

$\Delta;\Gamma, x : A \vdash e : B$ Subderivation

$\Delta;\Gamma, x : A \sqsubseteq \Delta';\Gamma', x : A$ Rule

Weakening Lemma

If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ and $\Delta;\Gamma \vdash e : A$ then $\Delta';\Gamma' \vdash e : A$

Case $$\frac{\Delta;\Gamma, x : A \vdash e : B}{\Delta;\Gamma \vdash \lambda x.e : A \to B}$$

$\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ Assumption

$\Delta;\Gamma, x : A \vdash e : B$ Subderivation

$\Delta;\Gamma, x : A \sqsubseteq \Delta';\Gamma', x : A$ Rule

$\Delta';\Gamma', x : A \vdash e : B$ Induction

Weakening Lemma

If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ and $\Delta;\Gamma \vdash e : A$ then $\Delta';\Gamma' \vdash e : A$

Case $\dfrac{\Delta;\Gamma, x : A \vdash e : B}{\Delta;\Gamma \vdash \lambda x.e : A \to B}$

$\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$	Assumption
$\Delta;\Gamma, x : A \vdash e : B$	Subderivation
$\Delta;\Gamma, x : A \sqsubseteq \Delta';\Gamma', x : A$	Rule
$\Delta';\Gamma', x : A \vdash e : B$	Induction
$\Delta';\Gamma' \vdash \lambda x.e : A \to B$	Rule

Weakening Lemma

If $\Delta; \Gamma \sqsubseteq \Delta'; \Gamma'$ and $\Delta; \Gamma \vdash e : A$ then $\Delta'; \Gamma' \vdash e : A$

Case $$\frac{\Delta; \Gamma \vdash e : \Diamond A \qquad \Delta; x : A \vdash e' : \Diamond B}{\Delta; \Gamma \vdash \mathsf{let}\ \mathsf{dia}(x) = e\ \mathsf{in}\ e' : \Diamond B}$$

Weakening Lemma

If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ and $\Delta ; \Gamma \vdash e : A$ then $\Delta' ; \Gamma' \vdash e : A$

Case $$\frac{\Delta ; \Gamma \vdash e : \Diamond A \qquad \Delta ; x : A \vdash e' : \Diamond B}{\Delta ; \Gamma \vdash \text{let } \mathrm{dia}(x) = e \text{ in } e' : \Diamond B}$$

$\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ Assumption

Weakening Lemma

If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ and $\Delta;\Gamma \vdash e : A$ then $\Delta';\Gamma' \vdash e : A$

Case $\dfrac{\Delta;\Gamma \vdash e : \Diamond A \qquad \Delta; x:A \vdash e' : \Diamond B}{\Delta;\Gamma \vdash \text{let } \mathrm{dia}(x) = e \text{ in } e' : \Diamond B}$

$\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ Assumption

$\Delta;\Gamma \vdash e : \Diamond A$ Subderivation

$\Delta; x:A \vdash e' : \Diamond B$ Subderivation

Weakening Lemma

If $\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$ and $\Delta ; \Gamma \vdash e : A$ then $\Delta' ; \Gamma' \vdash e : A$

Case $$\frac{\Delta ; \Gamma \vdash e : \Diamond A \qquad \Delta ; x : A \vdash e' : \Diamond B}{\Delta ; \Gamma \vdash \mathrm{let}\ \mathrm{dia}(x) = e \ \mathrm{in}\ e' : \Diamond B}$$

$\Delta ; \Gamma \sqsubseteq \Delta' ; \Gamma'$	Assumption
$\Delta ; \Gamma \vdash e : \Diamond A$	Subderivation
$\Delta ; x : A \vdash e' : \Diamond B$	Subderivation
$\Delta' ; \Gamma' \vdash e : \Diamond A$	Induction

Weakening Lemma

If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ and $\Delta;\Gamma \vdash e : A$ then $\Delta';\Gamma' \vdash e : A$

Case $$\frac{\Delta;\Gamma \vdash e : \Diamond A \qquad \Delta; x:A \vdash e' : \Diamond B}{\Delta;\Gamma \vdash \mathrm{let}\ \mathrm{dia}(x) = e\ \mathrm{in}\ e' : \Diamond B}$$

$\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$	Assumption
$\Delta;\Gamma \vdash e : \Diamond A$	Subderivation
$\Delta; x:A \vdash e' : \Diamond B$	Subderivation
$\Delta';\Gamma' \vdash e : \Diamond A$	Induction
$\Delta; x:A \sqsubseteq \Delta'; x:A$	Forgetting + Rule

Weakening Lemma

If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ and $\Delta;\Gamma \vdash e : A$ then $\Delta';\Gamma' \vdash e : A$

Case $\dfrac{\Delta;\Gamma \vdash e : \Diamond A \qquad \Delta; x:A \vdash e' : \Diamond B}{\Delta;\Gamma \vdash \text{let } \mathrm{dia}(x) = e \text{ in } e' : \Diamond B}$

$\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$	Assumption
$\Delta;\Gamma \vdash e : \Diamond A$	Subderivation
$\Delta; x:A \vdash e' : \Diamond B$	Subderivation
$\Delta';\Gamma' \vdash e : \Diamond A$	Induction
$\Delta; x:A \sqsubseteq \Delta'; x:A$	Forgetting + Rule
$\Delta'; x:A \vdash e' : \Diamond B$	Induction

Weakening Lemma

If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ and $\Delta;\Gamma \vdash e : A$ then $\Delta';\Gamma' \vdash e : A$

Case $\dfrac{\Delta;\Gamma \vdash e : \Diamond A \qquad \Delta; x : A \vdash e' : \Diamond B}{\Delta;\Gamma \vdash \mathsf{let}\ \mathsf{dia}(x) = e\ \mathsf{in}\ e' : \Diamond B}$

$\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$	Assumption
$\Delta;\Gamma \vdash e : \Diamond A$	Subderivation
$\Delta; x : A \vdash e' : \Diamond B$	Subderivation
$\Delta';\Gamma' \vdash e : \Diamond A$	Induction
$\Delta; x : A \sqsubseteq \Delta'; x : A$	Forgetting + Rule
$\Delta'; x : A \vdash e' : \Diamond B$	Induction
$\Delta';\Gamma' \vdash \mathsf{let}\ \mathsf{dia}(x) = e\ \mathsf{in}\ e' : \Diamond B$	Rule

Substitutions

$$\boxed{\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'}$$

$$\frac{}{\Delta ; \Gamma \vdash (\cdot ; \cdot) : \bullet ; \bullet}$$

$$\frac{\Delta ; \Gamma \vdash \delta ; \gamma : \Delta' ; \Gamma' \qquad \Delta ; \bullet \vdash e : A}{\Delta ; \Gamma \vdash (\delta, e/x) ; \gamma : (\Delta', x : A) ; \Gamma'}$$

$$\frac{\Delta ; \Gamma \vdash \delta ; \gamma : \Delta' ; \Gamma' \qquad \Delta ; \Gamma \vdash e : A}{\Delta ; \Gamma \vdash \delta ; (\gamma, e/x) : \Delta' ; (\Gamma', x : A)}$$

Applying a Substitution

Applying a Substitution

$$[\delta ; \gamma]\ x = \gamma(x)$$

Applying a Substitution

$$[\delta ; \gamma] \; x = \gamma(x)$$

$$[\delta ; \gamma] \; \underline{x} = \delta(x)$$

Applying a Substitution

$$[\delta;\gamma]\ x = \gamma(x)$$

$$[\delta;\gamma]\ \underline{x} = \delta(x)$$

$$[\delta;\gamma]\ e_1\ e_2 = ([\delta;\gamma]e_1)([\delta;\gamma]e_2)$$

Applying a Substitution

$$[\delta;\gamma]\ x = \gamma(x)$$

$$[\delta;\gamma]\ \underline{x} = \delta(x)$$

$$[\delta;\gamma]\ e_1\ e_2 = ([\delta;\gamma]e_1)([\delta;\gamma]e_2)$$

$$[\delta;\gamma]\ \lambda x.e = \lambda x.\ [\delta;(\gamma, x/x)]e$$

Applying a Substitution

$$[\delta ; \gamma]\ x = \gamma(x)$$

$$[\delta ; \gamma]\ \underline{x} = \delta(x)$$

$$[\delta ; \gamma]\ e_1\ e_2 = ([\delta;\gamma]e_1)([\delta;\gamma]e_2)$$

$$[\delta ; \gamma]\ \lambda x . e = \lambda x .\ [\delta ; (\gamma, x/x)] e$$

$$[\delta ; \gamma]\ \mathrm{box}(e) = \mathrm{box}([\delta ; \cdot]\ e)$$

Applying a Substitution

$$
\begin{aligned}
[\delta;\gamma]\ x &= \gamma(x) \\
[\delta;\gamma]\ \underline{x} &= \delta(x) \\
[\delta;\gamma]\ e_1\ e_2 &= ([\delta;\gamma]e_1)([\delta;\gamma]e_2) \\
[\delta;\gamma]\ \lambda x.\, e &= \lambda x.\ [\delta;(\gamma, x/x)]e \\
[\delta;\gamma]\ \mathrm{box}(e) &= \mathrm{box}([\delta;\cdot]\, e) \\
[\delta;\gamma]\ (\mathrm{let}\ \mathrm{dia}(x) = e\ \mathrm{in}\ e') &= \mathrm{let}\ \mathrm{dia}(x) = [\delta;\gamma]e \\
&\ \mathrm{in} \\
&\ [\delta;(x/x)]\, e'
\end{aligned}
$$

Forgetting and Substitution

If $\Delta; \Gamma \vdash (\delta; \gamma) : \Delta'; \Gamma'$ then $\Delta; \cdot \vdash (\delta; \cdot) : \Delta'; \cdot$

Forgetting and Substitution

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ then $\Delta ; \cdot \vdash (\delta ; \cdot) : \Delta' ; \cdot$

Proof: By induction on

$$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$$

Weakening + Substitution

If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ and $\Delta;\Gamma \vdash (\delta;\gamma) : \Delta'';\Gamma''$

then $\quad \Delta';\Gamma' \vdash (\delta;\gamma) : \Delta'';\Gamma''$

Weakening + Substitution

If $\Delta;\Gamma \sqsubseteq \Delta';\Gamma'$ and $\Delta;\Gamma \vdash (\delta;\gamma) : \Delta'';\Gamma''$

then $\Delta';\Gamma' \vdash (\delta;\gamma) : \Delta'';\Gamma''$

Proof: By structural induction on

$$\Delta;\Gamma \vdash (\delta;\gamma) : \Delta'';\Gamma''$$

Substitution Theorem

If $\Gamma; \Delta \vdash (\delta; \gamma) : \Gamma'; \Delta'$ and $\Gamma'; \Delta' \vdash e : A$

then $\Gamma'; \Delta' \vdash [\delta; \gamma] e : A$

Substitution Theorem

If $\Delta; \Gamma \vdash (\delta; \gamma) : \Delta'; \Gamma'$ and $\Delta'; \Gamma' \vdash e : A$

then $\Delta'; \Gamma' \vdash [\delta; \gamma]e : A$

<u>Proof:</u> By induction on the derivation of $\Delta'; \Gamma' \vdash e : A$

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case : $\dfrac{\Delta' ; \Gamma', x : A \vdash e : B}{\Delta' ; \Gamma' \vdash \lambda x . e : A \to B}$

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case : $$\frac{\Delta' ; \Gamma', x : A \vdash e : B}{\Delta' ; \Gamma' \vdash \lambda x . e : A \to B}$$

$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ Assumption

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case : $$\frac{\Delta' ; \Gamma', x : A \vdash e : B}{\Delta' ; \Gamma' \vdash \lambda x . e : A \to B}$$

$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ Assumption

$\Delta ; \Gamma \sqsubseteq \Delta ; \Gamma, x : A$ Reflexivity of weakening rule

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case : $$\frac{\Delta' ; \Gamma', x : A \vdash e : B}{\Delta' ; \Gamma' \vdash \lambda x . e : A \to B}$$

$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ — Assumption

$\Delta ; \Gamma \sqsubseteq \Delta ; \Gamma, x : A$ — Reflexivity of weakening + rule

$\Delta ; \Gamma, x : A \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ — Substitution + weakening

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case : $$\frac{\Delta' ; \Gamma', x : A \vdash e : B}{\Delta' ; \Gamma' \vdash \lambda x . e : A \to B}$$

$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ Assumption

$\Delta ; \Gamma \sqsubseteq \Delta ; \Gamma, x : A$ Reflexivity of weakening rule

$\Delta ; \Gamma, x : A \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ Substitution + weakening

$\Delta ; \Gamma, x : A \vdash (\delta ; (\gamma, x / x)) : \Delta' ; \Gamma', x : A$ Rule

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case :
$$\frac{\Delta' ; \Gamma', x : A \vdash e : B}{\Delta' ; \Gamma' \vdash \lambda x . e : A \to B}$$

$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ Assumption

$\Delta ; \Gamma \sqsubseteq \Delta ; \Gamma, x : A$ Reflexivity of weakening + rule

$\Delta ; \Gamma, x : A \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ Substitution + weakening Rule

$\Delta ; \Gamma, x : A \vdash (\delta ; (\gamma, x/x)) : \Delta' ; \Gamma', x : A$

$\Delta ; \Gamma, x : A \vdash [\delta ; (\gamma, x/x)] e : B$ Induction on subderivation

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case : $$\dfrac{\Delta' ; \Gamma', x : A \vdash e : B}{\Delta' ; \Gamma' \vdash \lambda x . e : A \to B}$$

$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$	Assumption
$\Delta ; \Gamma \sqsubseteq \Delta ; \Gamma, x : A$	Reflexivity of weakening + rule
$\Delta ; \Gamma, x : A \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$	Substitution + weakening
$\Delta ; \Gamma, x : A \vdash (\delta ; (\gamma, x/x)) : \Delta' ; \Gamma', x : A$	Rule
$\Delta ; \Gamma, x : A \vdash [\delta ; (\gamma, x/x)] e : B$	Induction on subderivation
$\Delta ; \Gamma \vdash \lambda x . [\delta ; (\gamma, (x/x))] : A \to B$	Rule

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case : $\dfrac{\Delta' ; \Gamma', x : A \vdash e : B}{\Delta' ; \Gamma' \vdash \lambda x . e : A \to B}$

$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$	Assumption
$\Delta ; \Gamma \sqsubseteq \Delta ; \Gamma, x : A$	Reflexivity of weakening + rule
$\Delta ; \Gamma, x : A \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$	Substitution + weakening
$\Delta ; \Gamma, x : A \vdash (\delta ; (\gamma, x/x)) : \Delta' ; \Gamma', x : A$	Rule
$\Delta ; \Gamma, x : A \vdash [\delta ; (\gamma, x/x)] e : B$	Induction on subderivation
$\Delta ; \Gamma \vdash \lambda x . [\delta ; (\gamma, (x/x))] : A \to B$	Rule
$\Delta ; \Gamma \vdash [\delta ; \gamma] (\lambda x . e) : A \to B$	Definition of substitution

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case $$\frac{\Delta' ; \cdot \vdash e : A}{\Delta' ; \Gamma' \vdash \mathsf{box}(e) : \Box A}$$

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case $$\frac{\Delta' ; \cdot \vdash e : A}{\Delta' ; \Gamma' \vdash \mathrm{box}(e) : \Box A}$$

$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ Assumption

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case $$\frac{\Delta' ; \cdot \vdash e : A}{\Delta' ; \Gamma' \vdash \mathrm{box}(e) : \Box A}$$

$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ Assumption

$\Delta ; \cdot \vdash (\delta ; \cdot) : \Delta' ; \cdot$ Forgetting

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case $\dfrac{\Delta' ; \cdot \vdash e : A}{\Delta' ; \Gamma' \vdash \mathrm{box}(e) : \Box A}$

$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ Assumption

$\Delta ; \cdot \vdash (\delta ; \cdot) : \Delta' ; \cdot$ Forgetting

$\Delta ; \cdot \vdash [\delta ; \cdot] e : A$ Induction

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case $\dfrac{\Delta' ; \cdot \vdash e : A}{\Delta' ; \Gamma' \vdash \mathrm{box}(e) : \Box A}$

$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ Assumption

$\Delta ; \cdot \vdash (\delta ; \cdot) : \Delta' ; \cdot$ Forgetting

$\Delta ; \cdot \vdash [\delta ; \cdot] e : A$ Induction

$\Delta ; \Gamma \vdash \mathrm{box}([\delta ; \cdot] e) : \Box A$ Rule

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case $\dfrac{\Delta' ; \cdot \vdash e : A}{\Delta' ; \Gamma' \vdash \mathrm{box}(e) : \Box A}$

$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$	Assumption
$\Delta ; \cdot \vdash (\delta ; \cdot) : \Delta' ; \cdot$	Forgetting
$\Delta ; \cdot \vdash [\delta ; \cdot] e : A$	Induction
$\Delta ; \Gamma \vdash \mathrm{box}([\delta ; \cdot] e) : \Box A$	Rule
$\Delta ; \Gamma \vdash [\delta ; \gamma]\, \mathrm{box}(e) : \Box A$	Definition of substitution

Substitution Theorem

If $\Delta; \Gamma \vdash (\delta;\gamma) : \Delta'; \Gamma'$ and $\Delta'; \Gamma' \vdash e : A$

then $\Delta'; \Gamma' \vdash [\delta;\gamma]e : A$

Case $\dfrac{\Delta';\Gamma' \vdash e : \Diamond A \qquad \Delta'; x:A \vdash e' : B}{\Delta';\Gamma' \vdash \text{let dia}(x) = e \text{ in } e' : \Diamond B}$

1. $\Delta; \Gamma \vdash (\delta;\gamma) : \Delta'; \Gamma'$ — Assumption
2. $\Delta'; \Gamma' \vdash e : \Diamond A$ — Subderivation
3. $\Delta'; x:A \vdash e' : \Diamond B$ — Subderivation
4. $\Delta; \Gamma \vdash [\delta;\gamma]e : \Diamond A$ — Induction on (1), (3)
5. $\Delta; \cdot \vdash (\delta; \cdot) : \Delta'; \cdot$ — Forgetting on (1)
6. $\Delta; \cdot \sqsubseteq \Delta; x:A$ — Reflexivity + Rule
7. $\Delta; x:A \vdash (\delta; \cdot) : \Delta'; \cdot$ — Weakening + Substitution on (5) & (6)
8. $\Delta; x:A \vdash (\delta; (x/x)) : \Delta'; x:A$ — Rule on (7)
9. $\Delta; x:A \vdash [\delta;(x/x)]e' : \Diamond B$ — Induction on (8), (3)
10. $\Delta; \Gamma \vdash \text{let dia}(x) = [\delta;\gamma]e \text{ in } [\delta;(x/x)]e' : \Diamond B$ — Rule on (4), (9)
11. $\Delta; \Gamma \vdash [\delta;\gamma](\text{let dia}(x) = e \text{ in } e') : \Diamond B$ — Definition of substitution

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case $$\frac{\Delta' ; \Gamma' \vdash e : \Diamond A \qquad \Delta' ; x : A \vdash e' : B}{\Delta' ; \Gamma' \vdash \mathrm{let}\ \mathrm{dia}(x) = e\ \mathrm{in}\ e' : \Diamond B}$$

Substitution Theorem

If $\Delta;\Gamma \vdash (\delta;\gamma) : \Delta';\Gamma'$ and $\Delta';\Gamma' \vdash e : A$

then $\Delta';\Gamma' \vdash [\delta;\gamma]e : A$

Case $$\frac{\Delta';\Gamma' \vdash e : \Diamond A \qquad \Delta'; x : A \vdash e' : B}{\Delta';\Gamma' \vdash \mathrm{let}\ \mathrm{dia}(x) = e\ \mathrm{in}\ e' : \Diamond B}$$

1. $\Delta;\Gamma \vdash (\delta;\gamma) : \Delta';\Gamma'$ Assumption

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta;\gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta;\gamma] e : A$

Case $$\frac{\Delta' ; \Gamma' \vdash e : \Diamond A \qquad \Delta' ; x : A \vdash e' : B}{\Delta' ; \Gamma' \vdash \text{let dia}(x) = e \text{ in } e' : \Diamond B}$$

1. $\Delta ; \Gamma \vdash (\delta;\gamma) : \Delta' ; \Gamma'$ Assumption
2. $\Delta' ; \Gamma' \vdash e : \Diamond A$ Subderivation
3. $\Delta' ; x : A \vdash e' : \Diamond B$ Subderivation

Substitution Theorem

If $\Delta; \Gamma \vdash (\delta; \gamma) : \Delta'; \Gamma'$ and $\Delta'; \Gamma' \vdash e : A$

then $\Delta'; \Gamma' \vdash [\delta; \gamma] e : A$

Case $$\frac{\Delta'; \Gamma' \vdash e : \Diamond A \qquad \Delta'; x : A \vdash e' : B}{\Delta'; \Gamma' \vdash \text{let } \mathrm{dia}(x) = e \text{ in } e' : \Diamond B}$$

1. $\Delta; \Gamma \vdash (\delta; \gamma) : \Delta'; \Gamma'$ — Assumption
2. $\Delta'; \Gamma' \vdash e : \Diamond A$ — Subderivation
3. $\Delta'; x : A \vdash e' : \Diamond B$ — Subderivation
4. $\Delta; \Gamma \vdash [\delta; \gamma] e : \Diamond A$ — Induction on (1), (3)

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case $\dfrac{\Delta' ; \Gamma' \vdash e : \Diamond A \qquad \Delta' ; x : A \vdash e' : B}{\Delta' ; \Gamma' \vdash \text{let } \mathrm{dia}(x) = e \text{ in } e' : \Diamond B}$

1.	$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$	Assumption
2.	$\Delta' ; \Gamma' \vdash e : \Diamond A$	Subderivation
3.	$\Delta' ; x : A \vdash e' : \Diamond B$	Subderivation
4.	$\Delta ; \Gamma \vdash [\delta ; \gamma] e : \Diamond A$	Induction on (1), (3)
5.	$\Delta ; \cdot \vdash (\delta ; \cdot) : \Delta' ; \cdot$	Forgetting on (1)

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case $\dfrac{\Delta' ; \Gamma' \vdash e : \Diamond A \qquad \Delta' ; x : A \vdash e' : B}{\Delta' ; \Gamma' \vdash \text{let dia}(x) = e \text{ in } e' : \Diamond B}$

1.	$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$	Assumption
2.	$\Delta' ; \Gamma' \vdash e : \Diamond A$	Subderivation
3.	$\Delta' ; x : A \vdash e' : \Diamond B$	Subderivation
4.	$\Delta ; \Gamma \vdash [\delta ; \gamma] e : \Diamond A$	Induction on (1), (3)
5.	$\Delta ; \cdot \vdash (\delta ; \cdot) : \Delta' ; \cdot$	Forgetting on (1)
6.	$\Delta ; \cdot \sqsubseteq \Delta ; x : A$	Reflexivity + Rule

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case $\dfrac{\Delta' ; \Gamma' \vdash e : \Diamond A \qquad \Delta' ; x : A \vdash e' : B}{\Delta' ; \Gamma' \vdash \text{let dia}(x) = e \text{ in } e' : \Diamond B}$

1. $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ — Assumption
2. $\Delta' ; \Gamma' \vdash e : \Diamond A$ — Subderivation
3. $\Delta' ; x : A \vdash e' : \Diamond B$ — Subderivation
4. $\Delta ; \Gamma \vdash [\delta ; \gamma] e : \Diamond A$ — Induction on (1), (3)
5. $\Delta ; \cdot \vdash (\delta ; \cdot) : \Delta' ; \cdot$ — Forgetting on (1)
6. $\Delta ; \cdot \sqsubseteq \Delta ; x : A$ — Reflexivity + Rule
7. $\Delta ; x : A \vdash (\delta ; \cdot) : \Delta' ; \cdot$ — Weakening + Substitution on (5) & (6)

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta;\gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta;\gamma] e : A$

Case $\dfrac{\Delta' ; \Gamma' \vdash e : \Diamond A \qquad \Delta' ; x : A \vdash e' : B}{\Delta' ; \Gamma' \vdash \text{let } \mathrm{dia}(x) = e \text{ in } e' : \Diamond B}$

1.	$\Delta ; \Gamma \vdash (\delta;\gamma) : \Delta' ; \Gamma'$	Assumption
2.	$\Delta' ; \Gamma' \vdash e : \Diamond A$	Subderivation
3.	$\Delta' ; x : A \vdash e' : \Diamond B$	Subderivation
4.	$\Delta ; \Gamma \vdash [\delta;\gamma] e : \Diamond A$	Induction on (1), (3)
5.	$\Delta ; \cdot \vdash (\delta ; \cdot) : \Delta' ; \cdot$	Forgetting on (1)
6.	$\Delta ; \cdot \sqsubseteq \Delta ; x : A$	Reflexivity + Rule
7.	$\Delta ; x : A \vdash (\delta ; \cdot) : \Delta' ; \cdot$	Weakening + Substitution on (5) & (6)
8.	$\Delta ; x : A \vdash (\delta ; (x/x)) : \Delta' ; x : A$	Rule on (7)

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case $\dfrac{\Delta' ; \Gamma' \vdash e : \Diamond A \qquad \Delta' ; x : A \vdash e' : B}{\Delta' ; \Gamma' \vdash \text{let } \mathrm{dia}(x) = e \text{ in } e' : \Diamond B}$

1.	$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$	Assumption
2.	$\Delta' ; \Gamma' \vdash e : \Diamond A$	Subderivation
3.	$\Delta' ; x : A \vdash e' : \Diamond B$	Subderivation
4.	$\Delta ; \Gamma \vdash [\delta ; \gamma] e : \Diamond A$	Induction on (1), (3)
5.	$\Delta ; \cdot \vdash (\delta ; \cdot) : \Delta' ; \cdot$	Forgetting on (1)
6.	$\Delta ; \cdot \sqsubseteq \Delta ; x : A$	Reflexivity + Rule
7.	$\Delta ; x : A \vdash (\delta ; \cdot) : \Delta' ; \cdot$	Weakening + Substitution on (5) & (6)
8.	$\Delta ; x : A \vdash (\delta ; (x/x)) : \Delta' ; x : A$	Rule on (7)
9.	$\Delta ; x : A \vdash [\delta ; (x/x)] e' : \Diamond B$	Induction on (8), (3)

Substitution Theorem

If $\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$ and $\Delta' ; \Gamma' \vdash e : A$

then $\Delta' ; \Gamma' \vdash [\delta ; \gamma] e : A$

Case $$\frac{\Delta' ; \Gamma' \vdash e : \Diamond A \qquad \Delta' ; x : A \vdash e' : B}{\Delta' ; \Gamma' \vdash \mathrm{let}\ \mathrm{dia}(x) = e\ \mathrm{in}\ e' : \Diamond B}$$

1.	$\Delta ; \Gamma \vdash (\delta ; \gamma) : \Delta' ; \Gamma'$	Assumption
2.	$\Delta' ; \Gamma' \vdash e : \Diamond A$	Subderivation
3.	$\Delta' ; x : A \vdash e' : \Diamond B$	Subderivation
4.	$\Delta ; \Gamma \vdash [\delta ; \gamma] e : \Diamond A$	Induction on (1), (3)
5.	$\Delta ; \cdot \vdash (\delta ; \cdot) : \Delta' ; \cdot$	Forgetting on (1)
6.	$\Delta ; \cdot \sqsubseteq \Delta ; x : A$	Reflexivity + Rule
7.	$\Delta ; x : A \vdash (\delta ; \cdot) : \Delta' ; \cdot$	Weakening + Substitution on (5) & (6)
8.	$\Delta ; x : A \vdash (\delta ; (x/x)) : \Delta' ; x : A$	Rule on (7)
9.	$\Delta ; x : A \vdash [\delta ; (x/x)] e' : \Diamond B$	Induction on (8), (3)
10.	$\Delta ; \Gamma \vdash \mathrm{let}\ \mathrm{dia}(x) = [\delta ; \gamma] e\ \mathrm{in}\ [\delta ; (x/x)] e' : \Diamond B$	Rule on (4), (9)

Substitution Theorem

If $\Delta;\Gamma \vdash (\delta;\gamma) : \Delta';\Gamma'$ and $\Delta';\Gamma' \vdash e : A$

then $\Delta';\Gamma' \vdash [\delta;\gamma]e : A$

Case $$\frac{\Delta';\Gamma' \vdash e : \Diamond A \qquad \Delta'; x:A \vdash e' : B}{\Delta';\Gamma' \vdash \mathrm{let}\ \mathrm{dia}(x) = e\ \mathrm{in}\ e' : \Diamond B}$$

1.	$\Delta;\Gamma \vdash (\delta;\gamma) : \Delta';\Gamma'$	Assumption
2.	$\Delta';\Gamma' \vdash e : \Diamond A$	Subderivation
3.	$\Delta'; x:A \vdash e' : \Diamond B$	Subderivation
4.	$\Delta;\Gamma \vdash [\delta;\gamma]e : \Diamond A$	Induction on (1),(3)
5.	$\Delta;\cdot \vdash (\delta;\cdot) : \Delta';\cdot$	Forgetting on (1)
6.	$\Delta;\cdot \sqsubseteq \Delta; x:A$	Reflexivity + Rule
7.	$\Delta; x:A \vdash (\delta;\cdot) : \Delta';\cdot$	Weakening + Substitution on (5),(6)
8.	$\Delta; x:A \vdash (\delta; (x/x)) : \Delta'; x:A$	Rule on (7)
9.	$\Delta; x:A \vdash [\delta;(x/x)]e' : \Diamond B$	Induction on (8), (3)
10.	$\Delta;\Gamma \vdash \mathrm{let}\ \mathrm{dia}(x) = [\delta;\gamma]e\ \mathrm{in}\ [\delta;(x/x)]e' : \Diamond B$	Rule on (4), (9)
11.	$\Delta;\Gamma \vdash [\delta;\gamma](\mathrm{let}\ \mathrm{dia}(x) = e\ \mathrm{in}\ e') : \Diamond B$	Definition of substitution