Introduction to Logical Relations

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Simply-Typed Lambda Calculus

$$A ::= 1 \mid A \to B \mid A \times B$$

$$e ::= () \mid (e, e) \mid \pi_i(e) \mid \lambda x . e \mid e\ e \mid x$$

$$v ::= () \mid (v, v) \mid \lambda x . e$$

$$\Gamma ::= \cdot \mid \Gamma, x : A$$

$$\boxed{\Gamma \vdash e : A} \qquad \boxed{e \rightsquigarrow e'}$$

Typing   Reduction

Typing

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{}{\Gamma \vdash () : 1}$$

$$\frac{\Gamma \vdash e_1 : A_1 \quad \Gamma \vdash e_2 : A_2}{\Gamma \vdash (e_1, e_2) : A_1 \times A_2} \qquad \frac{\Gamma \vdash e : A_1 \times A_2}{\Gamma \vdash \pi_i(e) : A_i}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x . e : A \to B} \qquad \frac{\Gamma \vdash e_1 : A \to B \quad \Gamma \vdash e_2 : A}{\Gamma \vdash e_1 \, e_2 : B}$$

Evaluation

$$\frac{}{(\lambda x.e)\,v \rightsquigarrow [v/x]e} \qquad \frac{}{\pi_i\,(v_1, v_2) \rightsquigarrow v_i}$$

$$\frac{e_1 \rightsquigarrow e_1'}{e_1\,e_2 \rightsquigarrow e_1'\,e_2} \qquad \frac{e_2 \rightsquigarrow e_2'}{v\,e_2 \rightsquigarrow v\,e_2'}$$

$$\frac{e_1 \rightsquigarrow e_1'}{(e_1, e_2) \rightsquigarrow (e_1', e_2)} \qquad \frac{e_2 \rightsquigarrow e_2'}{(v, e_2) \rightsquigarrow (v, e_2')}$$

Evaluation

$$\frac{}{(\lambda x.e)\, v \leadsto [v/x]\, e} \qquad \frac{}{\pi_i\, (v_1, v_2) \leadsto v_i}$$

$$\frac{e_1 \leadsto e_1'}{e_1\, e_2 \leadsto e_1'\, e_2} \qquad \frac{e_2 \leadsto e_2'}{v\, e_2 \leadsto v\, e_2'}$$

Note that evaluation is deterministic

$$\frac{e_1 \leadsto e_1'}{(e_1, e_2) \leadsto (e_1', e_2)} \qquad \frac{e_2 \leadsto e_2'}{(v, e_2) \leadsto (v, e_2')}$$

Type Safety

Progress

If $\cdot \vdash e : A$ then $e \rightsquigarrow e'$ or e value

Preservation

If $\cdot \vdash e : A$ and $e \rightsquigarrow e'$ then $\cdot \vdash e' : A$

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* \checkmark$

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* \checkmark$

$\boxed{\text{We CANNOT prove this by induction}}$

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* \checkmark$

Case $\dfrac{\cdot \vdash e_1 : A \to B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case $\dfrac{\cdot \vdash e_1 : A \to B \qquad \cdot \vdash e_2 : B}{\cdot \vdash e_1\, e_2 : B}$

1. $\cdot \vdash e_1 : A \to B$ Subderivation

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* \checkmark$

Case $$\frac{\cdot \vdash e_1 : A \to B \qquad \cdot \vdash e_2 : B}{\cdot \vdash e_1\, e_2 : B}$$

1. $\cdot \vdash e_1 : A \to B$ Subderivation
2. $\cdot \vdash e_2 : A$ Subderivation

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* \checkmark$

Case $$\frac{\cdot \vdash e_1 : A \to B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1\, e_2 : B}$$

1. $\cdot \vdash e_1 : A \to B$ Subderivation
2. $\cdot \vdash e_2 : A$ Subderivation
3. $e_1 \leadsto^* v_1$ Induction

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* \checkmark$

Case $\dfrac{\cdot \vdash e_1 : A \to B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1 e_2 : B}$

1.	$\cdot \vdash e_1 : A \to B$	Subderivation
2.	$\cdot \vdash e_2 : A$	Subderivation
3.	$e_1 \leadsto^* v_1$	Induction
4.	$\cdot \vdash v_1 : A \to B$	Type Safety

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* \checkmark$

Case $\dfrac{\cdot \vdash e_1 : A \to B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1\, e_2 : B}$

1. $\cdot \vdash e_1 : A \to B$ — Subderivation
2. $\cdot \vdash e_2 : A$ — Subderivation
3. $e_1 \leadsto^* v_1$ — Induction
4. $\cdot \vdash v_1 : A \to B$ — Type Safety
5. $v_1 = \lambda x.\, e$ — Inversion

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case $\dfrac{\cdot \vdash e_1 : A \to B \qquad \cdot \vdash e_2 : B}{\cdot \vdash e_1\, e_2 : B}$

1. $\cdot \vdash e_1 : A \to B$ Subderivation
2. $\cdot \vdash e_2 : A$ Subderivation
3. $e_1 \rightsquigarrow^* v_1$ Induction
4. $\cdot \vdash v_1 : A \to B$ Type Safety
5. $v_1 = \lambda x . e$ Inversion
6. $x : A \vdash e : B$ Inversion

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* \checkmark$

Case $\dfrac{\cdot \vdash e_1 : A \to B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1\, e_2 : B}$

1. $\cdot \vdash e_1 : A \to B$ Subderivation
2. $\cdot \vdash e_2 : A$ Subderivation
3. $e_1 \rightsquigarrow^* v_1$ Induction
4. $\cdot \vdash v_1 : A \to B$ Type Safety
5. $v_1 = \lambda x . e$ Inversion
6. $x : A \vdash e : B$ Inversion
7. $e_2 \rightsquigarrow^* v_2$ Induction

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* \checkmark$

Case $$\frac{\cdot \vdash e_1 : A \to B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1\, e_2 : B}$$

1.	$\cdot \vdash e_1 : A \to B$	Subderivation
2.	$\cdot \vdash e_2 : A$	Subderivation
3.	$e_1 \leadsto^* v_1$	Induction
4.	$\cdot \vdash v_1 : A \to B$	Type Safety
5.	$v_1 = \lambda x . e$	Inversion
6.	$x : A \vdash e : B$	Inversion
7.	$e_2 \leadsto^* v_2$	Induction
8.	$\cdot \vdash v_2 : A$	Type Safety

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* \checkmark$

Case $\dfrac{\cdot \vdash e_1 : A \to B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1\, e_2 : B}$

1.	$\cdot \vdash e_1 : A \to B$	Subderivation
2.	$\cdot \vdash e_2 : A$	Subderivation
3.	$e_1 \leadsto^* v_1$	Induction
4.	$\cdot \vdash v_1 : A \to B$	Type Safety
5.	$v_1 = \lambda x . e$	Inversion
6.	$x : A \vdash e : B$	Inversion
7.	$e_2 \leadsto^* v_2$	Induction
8.	$\cdot \vdash v_2 : A$	Type Safety
9.	$\cdot \vdash [v_2/x]e : B$	Substitution

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* \checkmark$

Case $\dfrac{\cdot \vdash e_1 : A \to B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1\, e_2 : B}$

1.	$\cdot \vdash e_1 : A \to B$	Subderivation
2.	$\cdot \vdash e_2 : A$	Subderivation
3.	$e_1 \leadsto^* v_1$	Induction
4.	$\cdot \vdash v_1 : A \to B$	Type Safety
5.	$v_1 = \lambda x. e$	Inversion
6.	$x : A \vdash e : B$	Inversion
7.	$e_2 \leadsto^* v_2$	Induction
8.	$\cdot \vdash v_2 : A$	Type Safety
9.	$\cdot \vdash [v_2/x]e : B$	Substitution
10.	$[v_2/x]e \leadsto^* \checkmark$	

Termination

If $\cdot \vdash e : A$ then $e \rightsquigarrow^* v$

Case $\dfrac{\cdot \vdash e_1 : A \to B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1\, e_2 : B}$

1. $\cdot \vdash e_1 : A \to B$ — Subderivation
2. $\cdot \vdash e_2 : A$ — Subderivation
3. $e_1 \rightsquigarrow^* v_1$ — Induction
4. $\cdot \vdash v_1 : A \to B$ — Type Safety
5. $v_1 = \lambda x . e$ — Inversion
6. $x : A \vdash e : B$ — Inversion
7. $e_2 \rightsquigarrow^* v_2$ — Induction
8. $\cdot \vdash v_2 : A$ — Type Safety
9. $\cdot \vdash [v_2/x]e : B$ — Substitution
10. $[v_2/x]e \rightsquigarrow^* v$ — NOT BY INDUCTION

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* v$

Case $\dfrac{\cdot \vdash e_1 : A \to B \quad \cdot \vdash e_2 : B}{\cdot \vdash e_1\, e_2 : B}$

1.	$\cdot \vdash e_1 : A \to B$	Subderivation
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5.	$v_1 = \lambda x. e$	Inversion
6.	$x : A \vdash e : B$	Inversion
7.	$e_2 \leadsto^* v_2$	Induction
8.	$\cdot \vdash v_2 : A$	Type Safety
9.	$\cdot \vdash [v_2/x]e : B$	Substitution
10.	$[v_2/x]e \leadsto^* v$	NOT BY INDUCTION

1. $[v_2/x]e$ is not a subterm of $e_1\, e_2$!

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* v$

Case $\dfrac{\cdot \vdash e_1 : A \to B \qquad \cdot \vdash e_2 : B}{\cdot \vdash e_1\, e_2 : B}$

1. $\cdot \vdash e_1 : A \to B$	Subderivation
2. $\cdot \vdash e_2 : A$	Subderivation
3. $e_1 \leadsto^* v_1$	Induction
4. $\cdot \vdash v_1 : A \to B$	Type Safety
5. $v_1 = \lambda x . e$	Inversion
6. $x : A \vdash e : B$	Inversion
7. $e_2 \leadsto^* v_2$	Induction
8. $\cdot \vdash v_2 : A$	Type Safety
9. $\cdot \vdash [v_2/x]e : B$	Substitution
10. $[v_2/x]e \leadsto^* v$	NOT BY INDUCTION

1. $[v_2/x]\,e$ is not a subterm of $e_1\, e_2$!

2. We know nothing about e!

Termination

- knowing $e_1 \rightsquigarrow^* \lambda x.e$ doesn't tell us anything about $[v/x]e$
- We need to know applying v to $\lambda x.e$ terminates!

Defining a Logical Relation

We will define a type-indexed family of sets of terms

$$V_1 = \{ () \}$$

$$V_{A \times B} = \{ (v_1, v_2) \mid v_1 \in V_A \text{ and } v_2 \in V_B \}$$

$$V_{A \to B} = \{ v \mid \forall v' \in V_A .\ v\, v' \in \mathcal{E}_B \}$$

$$\mathcal{E}_A = \{ e \mid e \rightsquigarrow^* v \text{ and } v \in V_A \}$$

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$$\mathcal{E}_A = \{ e \mid e \leadsto^* v \text{ and } v \in V_A \}$$

one set V_A for each A

one set $\mathcal{E}_A$ for each A

$$\mathcal{E}_1 = \{ e \mid e \rightsquigarrow^* v,\ v \in V_1 \}$$
$$= \{ e \mid e \rightsquigarrow^* () \}$$

$$\mathcal{E}_{1 \to 1} = \{ e \mid e \rightsquigarrow^* f \text{ and } f \in V_{1 \to 1} \}$$
$$= \{ e \mid e \rightsquigarrow^* f \text{ and } \forall v' \in V_1 .\ f\, v' \in \mathcal{E}_1 \}$$
$$= \{ e \mid e \rightsquigarrow^* f \text{ and } f () \in \mathcal{E}_1 \}$$
$$= \{ e \mid e \rightsquigarrow^* f \text{ and } f() \rightsquigarrow^* () \}$$

$$\mathcal{E}_{(1 \to 1) \to 1}$$

Properties

1. for all A. $\quad V_A \subseteq \mathcal{E}_A$

Properties

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Proof: $\mathcal{E}_A = \{ e \mid e \rightsquigarrow^* v \wedge v \in V_A \}$

Since $v \rightsquigarrow^* v$ in 0 steps,

if $v \in V_A$ then $v \in \mathcal{E}_A$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

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Proof: Assume $e \rightsquigarrow e'$

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$\Leftarrow$: Assume $e' \in \mathcal{E}_A$

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Proof: Assume $e \rightsquigarrow e'$

$\Leftarrow$: Assume $e' \in \mathcal{E}_A$

$$\mathcal{E}_A = \{ e \mid e \rightsquigarrow^* v, \ v \in V_A \}$$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof: Assume $e \rightsquigarrow e'$

$\Leftarrow$: Assume $e' \in \mathcal{E}_A$

$\mathcal{E}_A = \{ e \mid e \rightsquigarrow^* v, v \in V_A \}$

Hence $e' \rightsquigarrow^* v$ and $v \in V_A$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof: Assume $e \rightsquigarrow e'$

$\Leftarrow$: Assume $e' \in \mathcal{E}_A$

$\mathcal{E}_A = \{ e \mid e \rightsquigarrow^* v, v \in V_A \}$

Hence $e' \rightsquigarrow^* v$ and $v \in V_A$

Since $e \rightsquigarrow e'$ and $e' \rightsquigarrow^* v$, $e \rightsquigarrow^* v$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof: Assume $e \rightsquigarrow e'$

$\Leftarrow$: Assume $e' \in \mathcal{E}_A$

$\mathcal{E}_A = \{ e \mid e \rightsquigarrow^* v,\ v \in V_A \}$

Hence $e' \rightsquigarrow^* v$ and $v \in V_A$

Since $e \rightsquigarrow e'$ and $e' \rightsquigarrow^* v$, $e \rightsquigarrow^* v$

So $e \rightsquigarrow^* v$ and $v \in V_A$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof: Assume $e \rightsquigarrow e'$

$\Leftarrow$: Assume $e' \in \mathcal{E}_A$

$\mathcal{E}_A = \{ e \mid e \rightsquigarrow^* v, v \in V_A \}$

Hence $e' \rightsquigarrow^* v$ and $v \in V_A$

Since $e \rightsquigarrow e'$ and $e' \rightsquigarrow^* v$, $e \rightsquigarrow^* v$

So $e \rightsquigarrow^* v$ and $v \in V_A$

So $e \in \mathcal{E}_A$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof: Assume $e \rightsquigarrow e'$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof: Assume $e \rightsquigarrow e'$

$\Rightarrow$: Assume $e \in \mathcal{E}_A$ $\qquad \mathcal{E}_A = \{ e \mid e \rightsquigarrow^* v, v \in V_A \}$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof: Assume $e \rightsquigarrow e'$

$\Rightarrow$: Assume $e \in \mathcal{E}_A$ $\qquad$ $\mathcal{E}_A = \{ e \mid e \rightsquigarrow^* v, \; v \in V_A \}$

Hence $e \rightsquigarrow^* v$ and $v \in V_A$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof: Assume $e \rightsquigarrow e'$

$\Rightarrow$: Assume $e \in \mathcal{E}_A$ $\qquad \mathcal{E}_A = \{e \mid e \rightsquigarrow^* v, v \in V_A\}$

Hence $e \rightsquigarrow^* v$ and $v \in V_A$

Note $e \rightsquigarrow e'$ and $e \rightsquigarrow^* v$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof: Assume $e \rightsquigarrow e'$

$\Rightarrow$: Assume $e \in \mathcal{E}_A$ $\qquad \mathcal{E}_A = \{ e \mid e \rightsquigarrow^* v,\ v \in V_A \}$

Hence $e \rightsquigarrow^* v$ and $v \in V_A$

Note $e \rightsquigarrow e'$ and $e \rightsquigarrow^* v$

Since evaluation is deterministic, $e' \rightsquigarrow^* v$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof: Assume $e \rightsquigarrow e'$

$\Rightarrow$: Assume $e \in \mathcal{E}_A$ $\qquad \mathcal{E}_A = \{e \mid e \rightsquigarrow^* v,\ v \in V_A\}$

Hence $e \rightsquigarrow^* v$ and $v \in V_A$

Note $e \rightsquigarrow e'$ and $e \rightsquigarrow^* v$

Since evaluation is deterministic, $e' \rightsquigarrow^* v$

So $e' \rightsquigarrow^* v$ and $v \in V_A$

Properties: Closure

If $e \rightsquigarrow e'$ then $e \in \mathcal{E}_A$ iff $e' \in \mathcal{E}_A$

Proof: Assume $e \rightsquigarrow e'$

$\Rightarrow$: Assume $e \in \mathcal{E}_A$ $\qquad \mathcal{E}_A = \{ e \mid e \rightsquigarrow^* v, v \in V_A \}$

Hence $e \rightsquigarrow^* v$ and $v \in V_A$

Note $e \rightsquigarrow e'$ and $e \rightsquigarrow^* v$

Since evaluation is deterministic, $e' \rightsquigarrow^* v$

So $e' \rightsquigarrow^* v$ and $v \in V_A$

So $e' \in \mathcal{E}_A$

Fundamental Lemma

$\Gamma = x:1,\ y:1\to 1,\ z:1\times 1$

Define V_Γ as follows:

$$V_\Gamma = \{(v_1/x, v_2/y, v_3/z) \mid v_1 \in V_1, v_2 \in V_{1\to 1}, v_3 \in V_{1\times 1}\}.$$

$$V_\cdot = \{[\,]\}$$

$$V_{(\Gamma, x:A)} = \{(\gamma, v/x) \mid \gamma \in V_\Gamma \text{ and } v \in V_A\}$$

Fundamental Lemma

Define V_Γ as follows:

$$V_\cdot = \{ [\,] \}$$

$$V_{(\Gamma, x : A)} = \{ (\gamma, v/x) \mid \gamma \in V_\Gamma \text{ and } v \in V_A \}$$

Fundamental Lemma:

If $\Gamma \vdash e : A$ and $\gamma \in V_\Gamma$ then $[\gamma]e \in \mathcal{E}_A$

Fundamental Lemma

Define V_Γ as follows:

$$V_\cdot = \{[\,]\}$$

$$V_{(\Gamma, x : A)} = \{(\gamma, v/x) \mid \gamma \in V_\Gamma \text{ and } v \in V_A\}$$

Fundamental Lemma:

If $\Gamma \vdash e : A$ and $\gamma \in V_\Gamma$ then $[\gamma]e \in \mathcal{E}_A$

Proof: By induction on $\Gamma \vdash e : A$

Proof

Case $\dfrac{x : A \in \Gamma}{\Gamma \vdash x : A}$

$\gamma \in V_\Gamma$ By assumption

$\gamma(x) \in V_A$ By definition of V_Γ

$\gamma(x) \in \mathcal{E}_A$ Since $V_A \subseteq \mathcal{E}_A$

$[\gamma]x \in \mathcal{E}_A$ By definition of $[\gamma]e$

Proof

Case $\dfrac{}{\Gamma \vdash () : 1}$

$() \in V_1$ By definition of V_1

$() \in \mathcal{E}_1$ Since $V_1 \subseteq \mathcal{E}_1$

Proof

Case $$\frac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

1. $\gamma \in V_\Gamma$ — Assumption
2. $\Gamma \vdash e_1 : A$ — Subderivation
3. $\Gamma \vdash e_2 : B$ — Subderivation

Proof

Case $$\frac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

1. $\gamma \in V_\Gamma$ — Assumption
2. $\Gamma \vdash e_1 : A$ — Subderivation
3. $\Gamma \vdash e_2 : B$ — Subderivation
4. $[\gamma]e_1 \in \mathcal{E}_A$ — Induction on (1), (2)

Proof

Case $\dfrac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A$	Subderivation
3.	$\Gamma \vdash e_2 : B$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_A$	Induction on (1), (2)
5.	$[\gamma]e_2 \in \mathcal{E}_B$	Induction on (1), (3)

Proof

Case $\dfrac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$

1. $\gamma \in V_\Gamma$ — Assumption
2. $\Gamma \vdash e_1 : A$ — Subderivation
3. $\Gamma \vdash e_2 : B$ — Subderivation
4. $[\gamma] e_1 \in \mathcal{E}_A$ — Induction on (1), (2)
5. $[\gamma] e_2 \in \mathcal{E}_B$ — Induction on (1), (3)
6. $[\gamma] e_1 \rightsquigarrow^* v_1$

Proof

Case $$\frac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A$	Subderivation
3.	$\Gamma \vdash e_2 : B$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_A$	Induction on (1), (2)
5.	$[\gamma]e_2 \in \mathcal{E}_B$	Induction on (1), (3)
6.	$[\gamma]e_1 \rightsquigarrow^* v_1$	
7.	$v_1 \in V_A$	Definition of $\mathcal{E}_A$

Proof

Case $$\frac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A$	Subderivation
3.	$\Gamma \vdash e_2 : B$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_A$	Induction on (1), (2)
5.	$[\gamma]e_2 \in \mathcal{E}_B$	Induction on (1), (3)
6.	$[\gamma]e_1 \leadsto^* v_1$	
7.	$v_1 \in V_A$	Definition of $\mathcal{E}_A$
8.	$[\gamma]e_2 \leadsto^* v_2$	

Proof

Case $\dfrac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A$	Subderivation
3.	$\Gamma \vdash e_2 : B$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_A$	Induction on (1), (2)
5.	$[\gamma]e_2 \in \mathcal{E}_B$	Induction on (1), (3)
6.	$[\gamma]e_1 \rightsquigarrow^* v_1$	
7.	$v_1 \in V_A$	Definition of $\mathcal{E}_A$
8.	$[\gamma]e_2 \rightsquigarrow^* v_2$	
9.	$v_2 \in V_B$	Definition of $\mathcal{E}_B$

Proof

Case $\dfrac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A$	Subderivation
3.	$\Gamma \vdash e_2 : B$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_A$	Induction on (1), (2)
5.	$[\gamma]e_2 \in \mathcal{E}_B$	Induction on (1), (3)
6.	$[\gamma]e_1 \rightsquigarrow^* v_1$	
7.	$v_1 \in V_A$	Definition of $\mathcal{E}_A$
8.	$[\gamma]e_2 \rightsquigarrow^* v_2$	
9.	$v_2 \in V_B$	Definition of $\mathcal{E}_B$
10.	$([\gamma]e_1, [\gamma]e_2) \rightsquigarrow^* (v_1, [\gamma]e_2)$	Reduction rules on (6)

Proof

Case $\dfrac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A$	Subderivation
3.	$\Gamma \vdash e_2 : B$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_A$	Induction on (1), (2)
5.	$[\gamma]e_2 \in \mathcal{E}_B$	Induction on (1), (3)
6.	$[\gamma]e_1 \rightsquigarrow^* v_1$	
7.	$v_1 \in V_A$	Definition of $\mathcal{E}_A$
8.	$[\gamma]e_2 \rightsquigarrow^* v_2$	
9.	$v_2 \in V_B$	Definition of $\mathcal{E}_B$
10.	$([\gamma]e_1, [\gamma]e_2) \rightsquigarrow^* (v_1, [\gamma]e_2)$	Reduction rules on (6)
11.	$(v_1, [\gamma]e_2) \rightsquigarrow^* (v_1, v_2)$	Reduction rules on (8)

Proof

Case $\dfrac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A$	Subderivation
3.	$\Gamma \vdash e_2 : B$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_A$	Induction on (1), (2)
5.	$[\gamma]e_2 \in \mathcal{E}_B$	Induction on (1), (3)
6.	$[\gamma]e_1 \rightsquigarrow^* v_1$	
7.	$v_1 \in V_A$	Definition of $\mathcal{E}_A$
8.	$[\gamma]e_2 \rightsquigarrow^* v_2$	
9.	$v_2 \in V_B$	Definition of $\mathcal{E}_B$
10.	$([\gamma]e_1, [\gamma]e_2) \rightsquigarrow^* (v_1, [\gamma]e_2)$	Reduction rules on (6)
11.	$(v_1, [\gamma]e_2) \rightsquigarrow^* (v_1, v_2)$	Reduction rules on (8)
12.	$([\gamma]e_1, [\gamma]e_2) \rightsquigarrow^* (v_1, v_2)$	Transitivity of $\rightsquigarrow^*$ on (10), (11)

Proof

Case $\dfrac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A$	Subderivation
3.	$\Gamma \vdash e_2 : B$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_A$	Induction on (1), (2)
5.	$[\gamma]e_2 \in \mathcal{E}_B$	Induction on (1), (3)
6.	$[\gamma]e_1 \rightsquigarrow^* v_1$	
7.	$v_1 \in V_A$	Definition of $\mathcal{E}_A$
8.	$[\gamma]e_2 \rightsquigarrow^* v_2$	
9.	$v_2 \in V_B$	Definition of $\mathcal{E}_B$
10.	$([\gamma]e_1, [\gamma]e_2) \rightsquigarrow^* (v_1, [\gamma]e_2)$	Reduction rules on (6)
11.	$(v_1, [\gamma]e_2) \rightsquigarrow^* (v_1, v_2)$	Reduction rules on (8)
12.	$([\gamma]e_1, [\gamma]e_2) \rightsquigarrow^* (v_1, v_2)$	Transitivity of $\rightsquigarrow^*$ on (10), (11)
13.	$[\gamma](e_1, e_2) \rightsquigarrow^* (v_1, v_2)$	Definition of $[\gamma]e$ on (12)

Proof

Case $\dfrac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash (e_1, e_2) : A \times B}$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A$	Subderivation
3.	$\Gamma \vdash e_2 : B$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_A$	Induction on (1), (2)
5.	$[\gamma]e_2 \in \mathcal{E}_B$	Induction on (1), (3)
6.	$[\gamma]e_1 \rightsquigarrow^* v_1$	
7.	$v_1 \in V_A$	Definition of $\mathcal{E}_A$
8.	$[\gamma]e_2 \rightsquigarrow^* v_2$	
9.	$v_2 \in V_B$	Definition of $\mathcal{E}_B$
10.	$([\gamma]e_1, [\gamma]e_2) \rightsquigarrow^* (v_1, [\gamma]e_2)$	Reduction rules on (6)
11.	$(v_1, [\gamma]e_2) \rightsquigarrow^* (v_1, v_2)$	Reduction rules on (8)
12.	$([\gamma]e_1, [\gamma]e_2) \rightsquigarrow^* (v_1, v_2)$	Transitivity of $\rightsquigarrow^*$ on (10), (11)
13.	$[\gamma](e_1, e_2) \rightsquigarrow^* (v_1, v_2)$	Definition of $[\gamma]e$ on (12)
14.	$(v_1, v_2) \in V_{A \times B}$	Definition of $V_{A \times B}$
15.	$[\gamma](e_1, e_2) \in \mathcal{E}_{A \times B}$	Definition of $\mathcal{E}_{A \times B}$

Proof

Case $\dfrac{\Gamma, x:A \vdash e : B}{\Gamma \vdash \lambda x.e : A \to B}$

0. $\Gamma, x:A \vdash e : B$ Subderivation
1. $\gamma \in V_\Gamma$ Assumption

Proof

Case $\dfrac{\Gamma, x:A \vdash e : B}{\Gamma \vdash \lambda x.e : A \to B}$

0. $\Gamma, x:A \vdash e : B$ Subderivation
1. $\gamma \in V_\Gamma$ Assumption
2. $[\gamma]\lambda x.e = \lambda x.[\gamma]e$ Definition

Proof

Case $$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x. e : A \to B}$$

0. $\Gamma, x : A \vdash e : B$ Subderivation
1. $\gamma \in V_\Gamma$ Assumption
2. $[\gamma]\lambda x. e = \lambda x. [\gamma] e$ Definition

WTS : $\lambda x. [\gamma] e \in V_{A \to B}$

Proof

Case
$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x. e : A \to B}$$

0. $\Gamma, x : A \vdash e : B$ Subderivation
1. $\gamma \in V_\Gamma$ Assumption
2. $[\gamma]\lambda x.e = \lambda x.[\gamma]e$ Definition

WTS: $\lambda x.[\gamma]e \in V_{A \to B}$

3. Assume $v \in V_A$

Proof

Case $$\frac{\Gamma, x:A \vdash e : B}{\Gamma \vdash \lambda x.e : A \to B}$$

0. $\Gamma, x:A \vdash e : B$ Subderivation
1. $\gamma \in V_\Gamma$ Assumption
2. $[\gamma]\lambda x.e = \lambda x.[\gamma]e$ Definition

WTS : $\lambda x.[\gamma]e \in V_{A \to B}$

3. Assume $v \in V_A$

   WTS $(\lambda x.[\gamma]e)\, v \in \mathcal{E}_B$

Proof

Case
$$\frac{\Gamma, x:A \vdash e : B}{\Gamma \vdash \lambda x.e : A \to B}$$

0. $\Gamma, x:A \vdash e : B$ Subderivation
1. $\gamma \in V_\Gamma$ Assumption
2. $[\gamma]\lambda x.e = \lambda x.[\gamma]e$ Definition

WTS : $\lambda x.[\gamma]e \in V_{A\to B}$

3. Assume $v \in V_A$

   WTS $(\lambda x.[\gamma]e)\,v \in \mathcal{E}_B$

   4. $(\lambda x.[\gamma]e)\,v \rightsquigarrow [\gamma, v/x]e$ Reduction

Proof

Case
$$\frac{\Gamma, x:A \vdash e : B}{\Gamma \vdash \lambda x.e : A \to B}$$

0. $\Gamma, x:A \vdash e : B$ Subderivation
1. $\gamma \in V_\Gamma$ Assumption
2. $[\gamma]\lambda x.e = \lambda x.[\gamma]e$ Definition

WTS: $\lambda x.[\gamma]e \in V_{A \to B}$

3. Assume $v \in V_A$

   WTS $(\lambda x.[\gamma]e)\,v \in \mathcal{E}_B$

   4. $(\lambda x.[\gamma]e)\,v \rightsquigarrow [\gamma, v/x]e$ Reduction
   5. $[\gamma, v/x] \in V_{\Gamma, x:A}$ Definition of V_Γ

Proof

Case
$$\frac{\Gamma, x:A \vdash e : B}{\Gamma \vdash \lambda x.e : A \to B}$$

0. $\Gamma, x:A \vdash e : B$ — Subderivation
1. $\gamma \in V_\Gamma$ — Assumption
2. $[\gamma]\,\lambda x.e = \lambda x.\,[\gamma]e$ — Definition

WTS : $\lambda x.[\gamma]e \in V_{A \to B}$

3. Assume $v \in V_A$

   WTS $(\lambda x.[\gamma]e)\,v \in \mathcal{E}_B$

   4. $(\lambda x.[\gamma]e)\,v \rightsquigarrow [\gamma, v/x]e$ — Reduction
   5. $[\gamma, v/x] \in V_{\Gamma, x:A}$ — Definition of V_Γ
   6. $[\gamma, v/x]e \in \mathcal{E}_B$ — Induction on (0), (5)

Proof

Case $\dfrac{\Gamma, x:A \vdash e : B}{\Gamma \vdash \lambda x.e : A \to B}$

0. $\Gamma, x:A \vdash e : B$ — Subderivation
1. $\gamma \in V_\Gamma$ — Assumption
2. $[\gamma]\lambda x.e = \lambda x.[\gamma]e$ — Definition

WTS: $\lambda x.[\gamma]e \in V_{A \to B}$

3. Assume $v \in V_A$

   WTS $(\lambda x.[\gamma]e)\, v \in \mathcal{E}_B$

   4. $(\lambda x.[\gamma]e)\, v \rightsquigarrow [\gamma, v/x]e$ — Reduction
   5. $[\gamma, v/x] \in V_{\Gamma, x:A}$ — Definition of V_Γ
   6. $[\gamma, v/x]e \in \mathcal{E}_B$ — Induction on (0), (5)
   7. $(\lambda x.[\gamma]e)\, v \in \mathcal{E}_B$ — Closure on (4), (6)

Proof

Case $\dfrac{\Gamma, x:A \vdash e : B}{\Gamma \vdash \lambda x.e : A \to B}$

0. $\Gamma, x:A \vdash e : B$ Subderivation
1. $\gamma \in V_\Gamma$ Assumption
2. $[\gamma]\lambda x.e = \lambda x.[\gamma]e$ Definition

WTS: $\lambda x.[\gamma]e \in V_{A \to B}$

3. Assume $v \in V_A$

   WTS $(\lambda x.[\gamma]e)\,v \in \mathcal{E}_B$

   4. $(\lambda x.[\gamma]e)\,v \rightsquigarrow [\gamma, v/x]e$ Reduction
   5. $[\gamma, v/x] \in V_{\Gamma, x:A}$ Definition of V_Γ
   6. $[\gamma, v/x]e \in \mathcal{E}_B$ Induction on (0), (5)
   7. $(\lambda x.[\gamma]e)\,v \in \mathcal{E}_B$ Closure on (4), (6)

8. $\lambda x.[\gamma]e \in V_{A \to B}$ Definition of $V_{A \to B}$

Proof

Case $\dfrac{\Gamma, x{:}A \vdash e : B}{\Gamma \vdash \lambda x.e : A \to B}$

0. $\Gamma, x{:}A \vdash e : B$ Subderivation
1. $\gamma \in V_\Gamma$ Assumption
2. $[\gamma]\lambda x.e = \lambda x.[\gamma]e$ Definition

WTS : $\lambda x.[\gamma]e \in V_{A\to B}$

3. Assume $v \in V_A$

   WTS $(\lambda x.[\gamma]e)\,v \in \mathcal{E}_B$

   4. $(\lambda x.[\gamma]e)\,v \rightsquigarrow [\gamma, v/x]e$ Reduction
   5. $[\gamma, v/x] \in V_{\Gamma, x:A}$ Definition of V_Γ
   6. $[\gamma, v/x]e \in \mathcal{E}_B$ Induction on (0), (5)
   7. $(\lambda x.[\gamma]e)\,v \in \mathcal{E}_B$ Closure on (4), (6)

8. $\lambda x.[\gamma]e \in V_{A\to B}$ Definition of $V_{A\to B}$
9. $\lambda x.[\gamma]e \in \mathcal{E}_{A\to B}$ $V_{A\to B} \subseteq \mathcal{E}_{A\to B}$

Proof

Case
$$\frac{\Gamma \vdash e_1 : A \to B \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1\, e_2 : B}$$

1. $\gamma \in V_\Gamma$ — Assumption
2. $\Gamma \vdash e_1 : A \to B$ — Subderivation
3. $\Gamma \vdash e_2 : A$ — Subderivation

Proof

Case $\dfrac{\Gamma \vdash e_1 : A \to B \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1\ e_2 : B}$

1. $\gamma \in V_\Gamma$	Assumption
2. $\Gamma \vdash e_1 : A \to B$	Subderivation
3. $\Gamma \vdash e_2 : A$	Subderivation
4. $[\gamma] e_1 \in \mathcal{E}_{A \to B}$	Induction on 1, 2

Proof

Case $$\frac{\Gamma \vdash e_1 : A \to B \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1\ e_2 : B}$$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A \to B$	Subderivation
3.	$\Gamma \vdash e_2 : A$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_{A \to B}$	Induction on 1, 2
5.	$[\gamma]e_1 \rightsquigarrow^* v_1$, $v_1 \in V_{A \to B}$	Def. of $\mathcal{E}_{A \to B}$

Proof

Case $$\frac{\Gamma \vdash e_1 : A \to B \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1\ e_2 : B}$$

1. $\gamma \in V_\Gamma$	Assumption
2. $\Gamma \vdash e_1 : A \to B$	Subderivation
3. $\Gamma \vdash e_2 : A$	Subderivation
4. $[\gamma]e_1 \in \mathcal{E}_{A \to B}$	Induction on 1, 2
5. $[\gamma]e_1 \rightsquigarrow^* v_1$, $v_1 \in V_{A \to B}$	Def. of $\mathcal{E}_{A \to B}$
6. $[\gamma]e_2 \in \mathcal{E}_A$	Induction on 1, 3

Proof

Case $$\frac{\Gamma \vdash e_1 : A \to B \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1\ e_2 : B}$$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A \to B$	Subderivation
3.	$\Gamma \vdash e_2 : A$	Subderivation
4.	$[\gamma] e_1 \in \mathcal{E}_{A \to B}$	Induction on 1, 2
5.	$[\gamma] e_1 \rightsquigarrow^* v_1$, $v_1 \in V_{A \to B}$	Def. of $\mathcal{E}_{A \to B}$
6.	$[\gamma] e_2 \in \mathcal{E}_A$	Induction on 1, 3
7.	$[\gamma] e_2 \rightsquigarrow^* v_2$, $v_2 \in V_A$	Def. of $\mathcal{E}_A$

Proof

Case $\dfrac{\Gamma \vdash e_1 : A \to B \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1\ e_2 : B}$

1. $\gamma \in V_\Gamma$	Assumption
2. $\Gamma \vdash e_1 : A \to B$	Subderivation
3. $\Gamma \vdash e_2 : A$	Subderivation
4. $[\gamma]e_1 \in \mathcal{E}_{A \to B}$	Induction on 1, 2
5. $[\gamma]e_1 \rightsquigarrow^* v_1$, $v_1 \in V_{A \to B}$	Def. of $\mathcal{E}_{A \to B}$
6. $[\gamma]e_2 \in \mathcal{E}_A$	Induction on 1, 3
7. $[\gamma]e_2 \rightsquigarrow^* v_2$, $v_2 \in V_A$	Def. of $\mathcal{E}_A$
8. $[\gamma]e_1\ [\gamma]e_2 \rightsquigarrow^* v_1\ [\gamma]e_2$	Reduction rules on 5

Proof

Case $$\frac{\Gamma \vdash e_1 : A \to B \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1\ e_2 : B}$$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A \to B$	Subderivation
3.	$\Gamma \vdash e_2 : A$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_{A \to B}$	Induction on 1, 2
5.	$[\gamma]e_1 \rightsquigarrow^* v_1\ ,\ v_1 \in V_{A \to B}$	Def. of $\mathcal{E}_{A \to B}$
6.	$[\gamma]e_2 \in \mathcal{E}_A$	Induction on 1, 3
7.	$[\gamma]e_2 \rightsquigarrow^* v_2\ ,\ v_2 \in V_A$	Def. of $\mathcal{E}_A$
8.	$[\gamma]e_1\ [\gamma]e_2 \rightsquigarrow^* v_1\ [\gamma]e_2$	Reduction rules on 5
9.	$v_1\ [\gamma]e_2 \rightsquigarrow^* v_1\ v_2$	Reduction rules on 7

Proof

Case
$$\frac{\Gamma \vdash e_1 : A \to B \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1\ e_2 : B}$$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A \to B$	Subderivation
3.	$\Gamma \vdash e_2 : A$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_{A\to B}$	Induction on 1, 2
5.	$[\gamma]e_1 \leadsto^* v_1\ ,\ v_1 \in V_{A\to B}$	Def. of $\mathcal{E}_{A\to B}$
6.	$[\gamma]e_2 \in \mathcal{E}_A$	Induction on 1, 3
7.	$[\gamma]e_2 \leadsto^* v_2\ ,\ v_2 \in V_A$	Def. of $\mathcal{E}_A$
8.	$[\gamma]e_1\ [\gamma]e_2 \leadsto^* v_1\ [\gamma]e_2$	Reduction rules on 5
9.	$v_1\ [\gamma]e_2 \leadsto^* v_1\ v_2$	Reduction rules on 7
10.	$v_1\ v_2 \in \mathcal{E}_B$	Def of $V_{A\to B}$

Proof

Case $\dfrac{\Gamma \vdash e_1 : A \to B \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1\ e_2 : B}$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A \to B$	Subderivation
3.	$\Gamma \vdash e_2 : A$	Subderivation
4.	$[\gamma]e_1 \in \mathcal{E}_{A \to B}$	Induction on 1, 2
5.	$[\gamma]e_1 \rightsquigarrow^* v_1$, $v_1 \in V_{A \to B}$	Def. of $\mathcal{E}_{A \to B}$
6.	$[\gamma]e_2 \in \mathcal{E}_A$	Induction on 1, 3
7.	$[\gamma]e_2 \rightsquigarrow^* v_2$, $v_2 \in V_A$	Def. of $\mathcal{E}_A$
8.	$[\gamma]e_1\ [\gamma]e_2 \rightsquigarrow^* v_1\ [\gamma]e_2$	Reduction rules on 5
9.	$v_1\ [\gamma]e_2 \rightsquigarrow^* v_1\ v_2$	Reduction rules on 7
10.	$v_1\ v_2 \in \mathcal{E}_B$	Def of $V_{A \to B}$
11.	$[\gamma]e_1\ [\gamma]e_2 \in \mathcal{E}_B$	Closure x2

Proof

Case
$$\frac{\Gamma \vdash e_1 : A \to B \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1\ e_2 : B}$$

1.	$\gamma \in V_\Gamma$	Assumption
2.	$\Gamma \vdash e_1 : A \to B$	Subderivation
3.	$\Gamma \vdash e_2 : A$	Subderivation
4.	$[\gamma] e_1 \in \mathcal{E}_{A \to B}$	Induction on 1, 2
5.	$[\gamma] e_1 \leadsto^* v_1$, $v_1 \in V_{A \to B}$	Def. of $\mathcal{E}_{A\to B}$
6.	$[\gamma] e_2 \in \mathcal{E}_A$	Induction on 1, 3
7.	$[\gamma] e_2 \leadsto^* v_2$, $v_2 \in V_A$	Def. of $\mathcal{E}_A$
8.	$[\gamma]e_1\ [\gamma]e_2 \leadsto^* v_1\ [\gamma]e_2$	Reduction rules on 5
9.	$v_1\ [\gamma]e_2 \leadsto^* v_1\ v_2$	Reduction rules on 7
10.	$v_1\ v_2 \in \mathcal{E}_B$	Def of $V_{A\to B}$
11.	$[\gamma]e_1\ [\gamma]e_2 \in \mathcal{E}_B$	Closure x2
12.	$[\gamma](e_1\ e_2) \in \mathcal{E}_B$	Def. of $[\gamma]$

Termination

If $\cdot \vdash e : A$ then $e \leadsto^* v$

Proof:

1.	$[\,] \in V_\cdot$	Definition
2.	$[\,]e \in \mathcal{E}_A$	Fundamental Lemma
3.	$e \in \mathcal{E}_A$	Definition of $[\gamma]$
4.	$e \leadsto^* v$	Definition of $\mathcal{E}_A$