Automata Theory

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Finite Automata

Schematic of Finite Automata

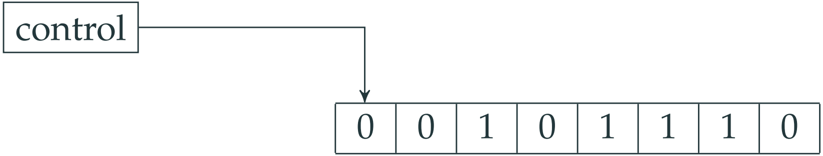


Figure 1: Schematic of Finite Automata

- A finite automaton has a finite set of control states.
- A finite automaton reads input symbols from left to right.
- A finite automaton accepts or rejects an input after reading the input.

Finite Automaton M_1

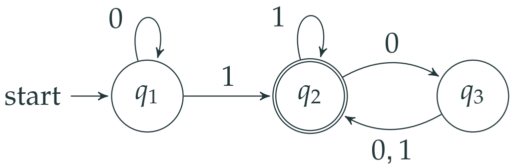
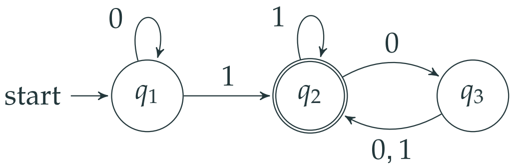


Figure 2: A Finite Automaton M_1

Figure 2 shows the state diagram of a finite automaton M_1. M_1 has

- 3 states: q_1, q_2, q_3;
- a start state: q_1;
- an accept state: q_2;
- 6 transitions: $q_1 \xrightarrow{0} q_1$, $q_1 \xrightarrow{1} q_2$, $q_2 \xrightarrow{1} q_2$, $q_2 \xrightarrow{0} q_3$, $q_3 \xrightarrow{0} q_2$, and $q_3 \xrightarrow{1} q_2$.

Accepted and Rejected String



- Consider an input string `1100`.
- M_1 processes the string from the start state q_1.
- It takes the transition labeled by the current symbol and moves to the next state.
- At the end of the string, there are two cases:
  - If M_1 is at an accept state, M_1 outputs accept;
  - Otherwise, M_1 outputs reject.
- Strings accepted by M_1: `1`, `01`, `11`, `1100`, `1101`, ...
- Strings rejected by M_1: `0`, `00`, `10`, `010`, `1010`, ...

Finite Automaton – Formal Definition

- A finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where
  - Q is a finite set of states;
  - Σ is a finite set called alphabet;
  - $\delta : Q \times \Sigma \rightarrow Q$ is the transition function;
  - $q_0 \in Q$ is the start state; and
  - $F \subseteq Q$ is the set of accept states.
- The set of strings accepted by M is called the language of machine M (written $L(M)$).
  - Hence a language is a set of strings.
- We also say M recognizes (or accepts) $L(M)$.

M_1 – Formal Definition

- The finite automaton $M_1 = (Q, \Sigma, \delta, q_1, F)$ consists of
  - $Q = \{q_1, q_2, q_3\}$;
  - $\Sigma = \{\texttt{0}, \texttt{1}\}$;
  - $\delta : Q \times \Sigma \to Q$ is

	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

  - q_1 is the start state; and
  - $F = \{q_2\}$.
- Moreover, we have

$$L(M_1) = \{w : \quad w \text{ contains at least one } \texttt{1} \text{ and an even number of } \texttt{0}\text{'s follow the last } \texttt{1}\}$$

Finite Automaton M_2

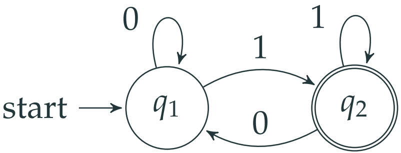


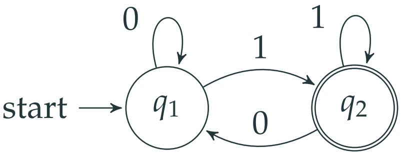
Figure 3: Finite Automaton M_2

- Figure 3 shows $M_2 = (\{q_1, q_2\}, \{0, 1\}, \delta, q_1, \{q_2\})$ where δ is

	0	1
q_1	q_1	q_2
q_2	q_1	q_2

- What is $L(M_2)$?
  - $L(M_2) = \{w : w \text{ ends in a } 1\}$.

Finite Automaton M_2



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  - $L(M_2) = \{w : w \text{ ends in a } \mathtt{1}\}$.

Finite Automaton M_3

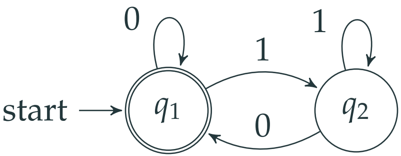


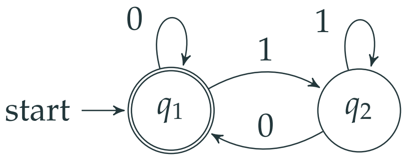
Figure 4: Finite Automaton M_3

- Figure 4 shows $M_3 = (\{q_1, q_2\}, \{0, 1\}, \delta, q_1, \{q_1\})$ where δ is

	0	1
q_1	q_1	q_2
q_2	q_1	q_2

- What is $L(M_3)$?
  - $L(M_3) = \{w : w$ is the empty string ϵ or ends in a $0\}$.

Finite Automaton M_3



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	0	1
q_1	q_1	q_2
q_2	q_1	q_2

- What is $L(M_3)$?
  - $L(M_3) = \{w : w$ is the empty string ϵ or ends in a $\mathtt{0}\}$.

Finite Automaton M_5

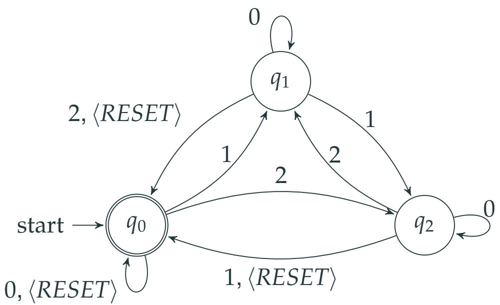


Figure 5: Finite Automaton M_5

- Figure 5 shows $M_5 = (\{q_0, q_1, q_2\}, \{\texttt{0}, \texttt{1}, \texttt{2}, \langle RESET \rangle\}, \delta, q_0, \{q_0\})$.
- Strings accepted by M_5: `0`, `00`, `12`, `21`, `012`, `102`, `120`, `021`, `201`, `210`, `111`, `222`, ...
- M_5 computes the sum of input symbols modulo 3. It resets upon the input symbol ⟨*RESET*⟩. Hence M_5 accepts strings whose sum is a multiple of 3 after ⟨*RESET*⟩.

Computation – Formal Definition

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton and $w = w_1 w_2 \cdots w_n$ a string where $w_i \in \Sigma$ for every $i = 1, \ldots, n$.
- We say M accepts w if there is a sequence of states $r_0, r_1, \ldots, r_n$ such that
  - $r_0 = q_0$;
  - $\delta(r_i, w_{i+1}) = r_{i+1}$ for $i = 0, \ldots, n-1$; and
  - $r_n \in F$.
- M recognizes language A if $A = \{w : M \text{ accepts } w\}$.

Definition 1
A language is called a regular language if some finite automaton recognizes it.

Regular Operations

Definition 2
Let A and B be languages. We define the following operations:

- Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- Concatenation: $A \circ B = \{xy : x \in A \text{ and } y \in B\}$.
- Star: $A^* = \{x_1 x_2 \cdots x_k : k \geq 0 \text{ and every } x_i \in A\}$.
- Complementation: $\overline{A} = \{x : x \in \Sigma^* \text{ but } x \notin A\}$.

- Note that $\epsilon \in A^*$ for every language A.

Closure Property – Union

Theorem 3
The class of regular languages is closed under the union operation. That is, $A_1 \cup A_2$ is regular if A_1 and A_2 are.

Proof.
Let $M_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$ recognize A_i for $i = 1, 2$. Construct $M = (Q, \Sigma, \delta, q_0, F)$ where

- $Q = Q_1 \times Q_2 = \{(r_1, r_2) : r_1 \in Q_1, r_2 \in Q_2\}$;
- $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$;
- $q_0 = (q_1, q_2)$;
- $F = (F_1 \times Q_2) \cup (Q_1 \times F_2) = \{(r_1, r_2) : r_1 \in F_1 \text{ or } r_2 \in F_2\}$. $\square$

- Why is $L(M) = A_1 \cup A_2$?

Nondeterminism

Nondeterminism

- When a machine is at a given state and reads an input symbol, there is precisely one choice of its next state.
- This is call deterministic computation.
- In nondeterministic machines, multiple choices may exist for the next state.
- A deterministic finite automaton is abbreviated as DFA; a nondeterministic finite automaton is abbreviated as NFA.
- A DFA is also an NFA.
- Since NFA allow more general computation, they can be much smaller than DFA.

NFA N_4

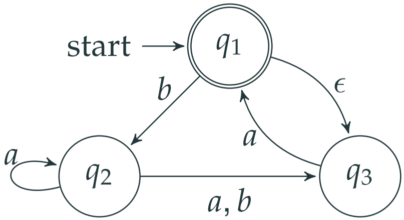


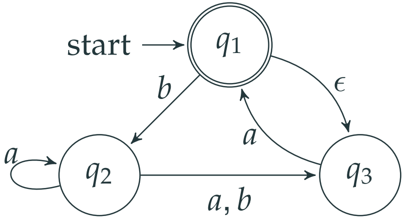
Figure 6: NFA N_4

- On input string `baa`, N_4 has several possible computation:
  - $q_1 \xrightarrow{\texttt{b}} q_2 \xrightarrow{\texttt{a}} q_2 \xrightarrow{\texttt{a}} q_2$;
  - $q_1 \xrightarrow{\texttt{b}} q_2 \xrightarrow{\texttt{a}} q_2 \xrightarrow{\texttt{a}} q_3$; or
  - $q_1 \xrightarrow{\texttt{b}} q_2 \xrightarrow{\texttt{a}} q_3 \xrightarrow{\texttt{a}} q_1$.

Nondeterministic Finite Automaton – Formal Definition

- For any set Q, $\mathcal{P}(Q) = \{R : R \subseteq Q\}$ denotes the power set of Q.
- For any alphabet Σ, define Σ_ϵ to be $\Sigma \cup \{\epsilon\}$.
- A nondeterministic finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where
  - Q is a finite set of states;
  - Σ is a finite alphabet;
  - $\delta : Q \times \Sigma_\epsilon \to \mathcal{P}(Q)$ is the transition function;
  - $q_0 \in Q$ is the start state; and
  - $F \subseteq Q$ is the accept states.
- Note that the transition function accepts the empty string as an input symbol.

NFA N_4 – Formal Definition



- $N_4 = (Q, \Sigma, \delta, q_1, \{q_1\})$ is a nondeterministic finite automaton where
  - $Q = \{q_1, q_2, q_3\}$;
  - Its transition function δ is

	ϵ	a	b
q_1	$\{q_3\}$	$\emptyset$	$\{q_2\}$
q_2	$\emptyset$	$\{q_2, q_3\}$	$\{q_3\}$
q_3	$\emptyset$	$\{q_1\}$	$\emptyset$

Nondeterministic Computation – Formal Definition

- Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA and w a string over Σ. We say N accepts w if w can be rewritten as $w = y_1 y_2 \cdots y_m$ with $y_i \in \Sigma_\epsilon$ and there is a sequence of states $r_0, r_1, \ldots, r_m$ such that
  - $r_0 = q_0$;
  - $r_{i+1} \in \delta(r_i, y_{i+1})$ for $i = 0, \ldots, m-1$; and
  - $r_m \in F$.
- Note that finitely many empty strings can be inserted in w.
- Also note that one sequence satisfying the conditions suffices to show the acceptance of an input string.
- Strings accepted by N_4: $\texttt{a}, \texttt{baa}, \ldots$.

Equivalence of NFA's and DFA's

Theorem 4
Every nondeterministic finite automaton has an equivalent deterministic finite automaton. That is, for every NFA N, there is a DFA M such that $L(M) = L(N)$.

Proof.
Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA. For $R \subseteq Q$, define $E(R) = \{q : q$ can be reached from R along 0 or more ϵ transitions $\}$. Construct a DFA $M = (Q', \Sigma, \delta', q'_0, F')$ where

- $Q' = \mathcal{P}(Q)$;
- $\delta'(R, a) = \{q \in Q : q \in E(\delta(r, a))$ for some $r \in R\}$;
- $q'_0 = E(\{q_0\})$;
- $F' = \{R \in Q' : R \cap F \neq \emptyset\}$.

□

- Why is $L(M) = L(N)$?

A DFA Equivalent to N_4

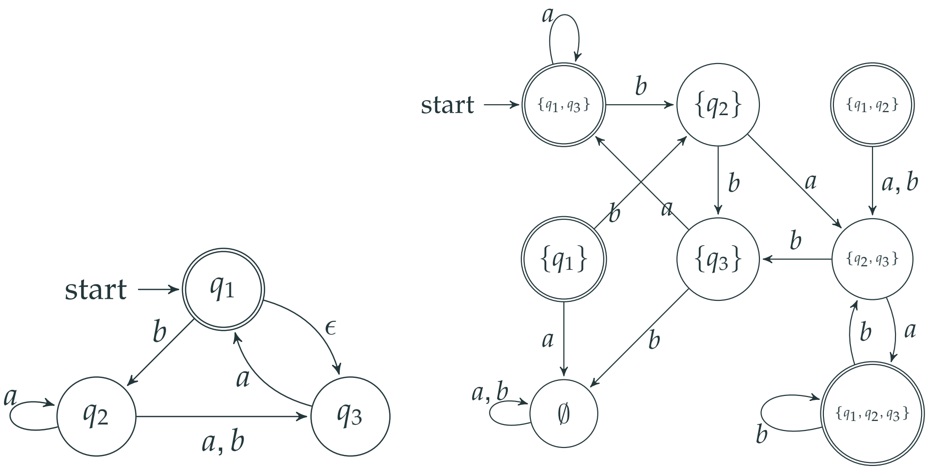


Figure 7: A DFA Equivalent to N_4

Closure Properties – Revisited

Theorem 5
The class of regular languages is closed under the union operation.

Proof.
Let $N_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$ recognize A_i for $i = 1, 2$. Construct $N = (Q, \Sigma, \delta, q_0, F)$ where

- $Q = \{q_0\} \cup Q_1 \cup Q_2$;
- $F = F_1 \cup F_2$; and
- $$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \\ \delta_2(q, a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \epsilon \\ \emptyset & q = q_0 \text{ and } a \neq \epsilon \end{cases}$$ □

- Why is $L(N) = L(N_1) \cup L(N_2)$?

Closure Properties – Revisited

Theorem 6
The class of regular languages is closed under the concatenation operation.

Proof.
Let $N_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$ recognize A_i for $i = 1, 2$. Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ where

- $Q = Q_1 \cup Q_2$; and
- $\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q, a) & q \in F_1 \text{ and } a \neq \epsilon \\ \delta_1(q, a) \cup \{q_2\} & q \in F_1 \text{ and } a = \epsilon \\ \delta_2(q, a) & q \in Q_2 \end{cases}$

□

- Why is $L(N) = L(N_1) \circ L(N_2)$?

Closure Properties – Revisited

Theorem 7
The class of regular languages is closed under the star operation.

Proof.
Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1. Construct $N = (Q, \Sigma, \delta, q_0, F)$ where

- $Q = \{q_0\} \cup Q_1$;
- $F = \{q_0\} \cup F_1$; and
- $\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q, a) & q \in F_1 \text{ and } a \neq \epsilon \\ \delta_1(q, a) \cup \{q_1\} & q \in F_1 \text{ and } a = \epsilon \\ \{q_1\} & q = q_0 \text{ and } a = \epsilon \\ \emptyset & q = q_0 \text{ and } a \neq \epsilon \end{cases}$

□

- Why is $L(N) = [L(N_1)]^*$?

Theorem 8
The class of regular languages is closed under complementation.

Proof.
Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing A. Consider $\overline{M} = (Q, \Sigma, \delta, q_0, Q \setminus F)$. We have $w \in L(M)$ if and only if $w \notin L(\overline{M})$. That is, $L(\overline{M}) = \overline{A}$ as required. □

Regular Expressions

Definition 9
R is a regular expression if R is

- a for some $a \in \Sigma$;
- ϵ;
- $\emptyset$;
- $(R_1 \cup R_2)$ where R_i's are regular expressions;
- $(R_1 \circ R_2)$ where R_i's are regular expressions; or
- (R_1^*) where R_1 is a regular expression.

Regular Expressions ii

- We write R^+ for $R \circ R^*$. Hence $R^* = R^+ \cup \epsilon$.
- Moreover, write R^k for $\overbrace{R \circ R \circ \cdots \circ R}^{k}$.
  - Define $R^0 = \epsilon$. We have $R^* = R^0 \cup R^1 \cup \cdots \cup R^n \cup \cdots$.
- $L(R)$ denotes the language described by the regular expression R.
- Note that $\emptyset \neq \{\epsilon\}$.

Examples of Regular Expressions

- For convenience, we write RS for $R \circ S$.
- We may also write the regular expression R to denote its language $L(R)$.
- $L(\mathtt{0}^*\mathtt{1}\mathtt{0}^*) = \{w : w \text{ contains a single } \mathtt{1}\}$.
- $L(\Sigma^*\mathtt{1}\Sigma^*) = \{w : w \text{ has at least one } \mathtt{1}\}$.
- $L((\Sigma\Sigma)^*) = \{w : w \text{ is a string of even length }\}$.
- $(\mathtt{0} \cup \epsilon)(1 \cup \epsilon) = \{\epsilon, \mathtt{0}, \mathtt{1}, \mathtt{01}\}$.
- $\mathtt{1}^*\emptyset = \emptyset$.
- $\emptyset^* = \{\epsilon\}$.
- For any regular expression R, we have $R \cup \emptyset = R$ and $R \circ \epsilon = R$.

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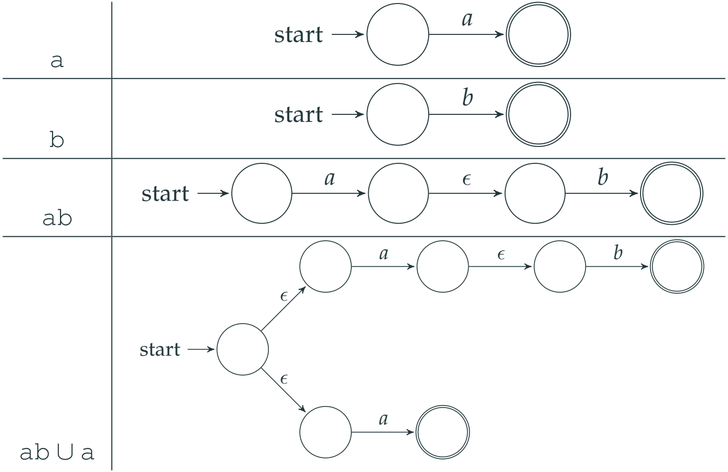
- For any regular expression R, we have $R \cup \emptyset = R$ and $R \circ \epsilon = R$.

Lemma 10
If a language is described by a regular expression, it is regular.

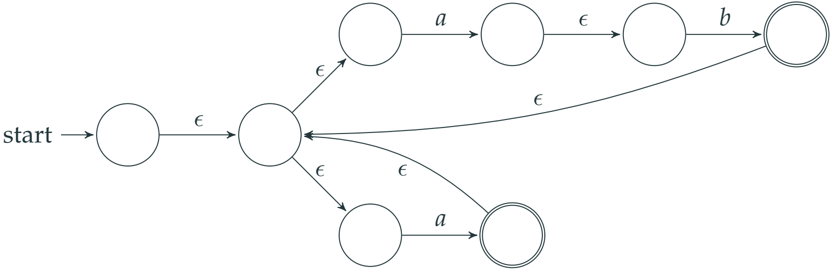
Proof.
We prove by induction on the regular expression R.

- $R = a$ for some $a \in \Sigma$. Consider the NFA $N_a = (\{q_1, q_2\}, \Sigma, \delta, q_1, \{q_2\})$ where
  $$\delta(r, y) = \begin{cases} \{q_2\} & r = q_1 \text{ and } y = a \\ \emptyset & \text{otherwise} \end{cases}$$
- $R = \epsilon$. Consider the NFA $N_\epsilon = (\{q_1\}, \Sigma, \delta, q_1, \{q_1\})$ where $\delta(r, y) = \emptyset$ for any r and y.
- $R = \emptyset$. Consider the NFA $N_\emptyset = (\{q_1\}, \Sigma, \delta, q_1, \emptyset)$ where $\delta(r, y) = \emptyset$ for any r and y.
- $R = R_1 \cup R_2$, $R = R_1 \circ R_2$, or $R = R_1^*$. By inductive hypothesis and the closure properties of finite automata. □

Regular Expressions and Finite Automata



$(\mathtt{ab} \cup \mathtt{a})^*$



Lemma 11
If a language is regular, it is described by a regular expression.

For the proof, we introduce a generalization of finite automata.

Definition 12
A generalized nondeterministic finite automaton is a 5-tuple $(Q, \Sigma, q_{\text{start}}, q_{\text{accept}})$ where

- Q is the finite set of states;
- Σ is the input alphabet;
- $\delta : Q \times Q \to \mathcal{R}$ is the transition function, where $\mathcal{R}$ denotes the set of regular expressions;
- q_{start} is the start state; and
- q_{accept} is the accept state.

A GNFA accepts a string $w \in \Sigma^*$ if $w = w_1 w_2 \cdots w_k$ where $w_i \in \Sigma^*$ and there is a sequence of states $r_0, r_1, \ldots, r_k$ such that

- $r_0 = q_{\text{start}}$;
- $r_k = q_{\text{accept}}$; and
- for every i, $w_i \in L(R_i)$ where $R_i = \delta(q_{i-1}, q_i)$.

Proof of Lemma.
Let M be the DFA for the regular language. Construct an equivalent GNFA G by adding $q_{\text{start}}, q_{\text{accept}}$ and necessary ϵ-transitions.

CONVERT (G):

1. Let k be the number of states of G.
2. If $k = 2$, then return the regular expression R labeling the transition from q_{start} to q_{accept}.
3. If $k > 2$, select $q_{\text{rip}} \in Q \setminus \{q_{\text{start}}, q_{\text{accept}}\}$. Construct $G' = (Q', \Sigma, \delta', q_{\text{start}}, q_{\text{accept}})$ where
   - $Q' = Q \setminus \{q_{\text{rip}}\}$;
   - for any $q_i \in Q' \setminus \{q_{\text{accept}}\}$ and $q_j \in Q' \setminus \{q_{\text{start}}\}$, define $\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup R_4$ where $R_1 = \delta(q_i, q_{\text{rip}})$, $R_2 = \delta(q_{\text{rip}}, q_{\text{rip}})$, $R_3 = \delta(q_{\text{rip}}, q_j)$, and $R_4 = \delta(q_i, q_j)$.
4. return CONVERT (G').

□

Lemma 13
For any GNFA G, CONVERT (G) is equivalent to G.

Proof.
We prove by induction on the number k of states of G.

- $k = 2$. Trivial.
- Assume the lemma holds for $k - 1$ states. We first show G' is equivalent to G. Suppose G accepts an input w. Let $q_{\text{start}}, q_1, q_2, \ldots, q_{\text{accept}}$ be an accepting computation of G. We have $q_{\text{start}} \xrightarrow{w_1} q_1 \cdots q_{i-1} \xrightarrow{w_i} q_i \xrightarrow{w_{i+1}} q_{\text{rip}} \cdots q_{\text{rip}} \xrightarrow{w_{j-1}} q_{\text{rip}} \xrightarrow{w_j} q_j \cdots q_{\text{accept}}$. Hence $q_{\text{start}} \xrightarrow{w_1} q_1 \cdots q_{i-1} \xrightarrow{w_i} q_i \xrightarrow{w_{i+1} \cdots w_j} q_j \cdots q_{\text{accept}}$ is a computation of G'. Conversely, any string accepted by G' is also accepted by G since the transition between q_i and q_j in G' describes the strings taking q_i to q_j in G. Hence G' is equivalent to G. By inductive hypothesis, CONVERT (G') is equivalent to G'. □

Regular Expressions and Finite Automata

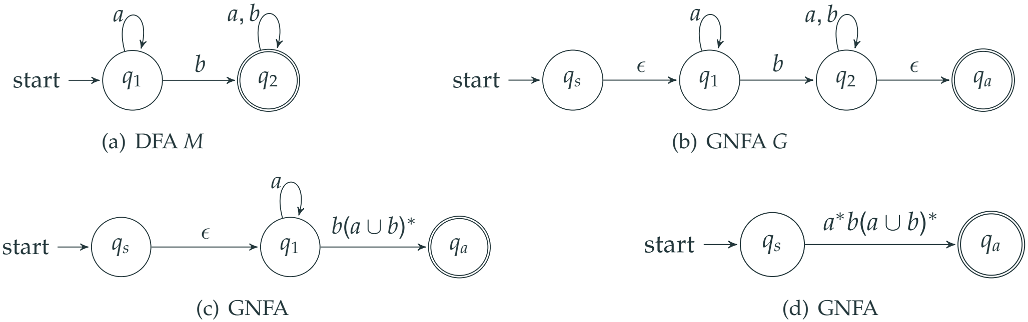


Figure 8: Finite Automaton to Regular Expression

Theorem 14
A language is regular if and only if some regular expression describes it.

Equivalence and Minimization

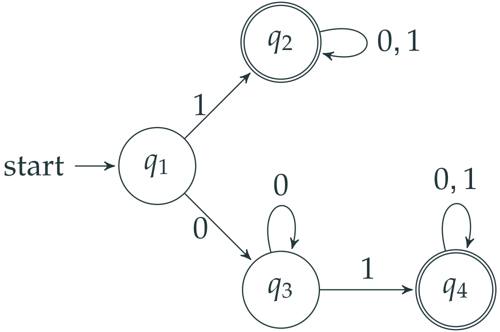
Equivalence of Descriptions

- Let M be a DFA, N an NFA, and R a regular expression.
- We would like to answer the following questions:
  - Is $L(M) = L(N)$?
  - Is $L(M) = L(R)$?
  - Is $L(N) = L(R)$?
- Recall that there are DFA's M_N and M_R such that $L(M_N) = L(N)$ and $L(M_R) = L(R)$.
- It suffices to solve the following problem:
  Given two DFA's M_0 and M_1, is $L(M_0) = L(M_1)$?

Equivalence of States

- Let us start with a simpler question.
- Give a DFA $M = (Q, \Sigma, \delta, q_0, F)$ and $p, q \in Q$, is it true that $p \xrightarrow{w} p' \in F$ if and only if $q \xrightarrow{w} q' \in F$ for all $w \in \Sigma^*$?
  - Note that p' need not be q'.
  - We only ask if p' and q' are both in F or not.
- If the answer is "yes," then p and q are equivalent.
- Otherwise, p and q are distinguishable.

Table-Filling Algorithm i

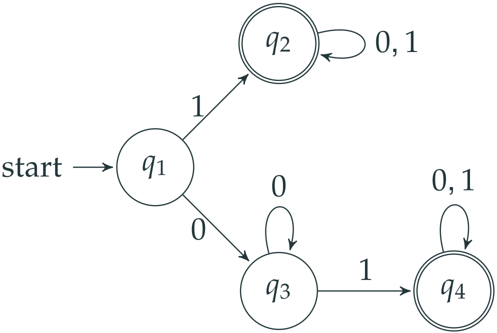


- Consider the DFA on the left.
- Since $q_1 \notin F$ but $q_2 \in F$, we know q_1 and q_2 are distinguishable.
  - Similarly, $\{q_1, q_4\}$, $\{q_3, q_2\}$, $\{q_3, q_4\}$ are all distinguishable.
  - Moreover, q_2 and q_4 all have self loops labeled by 0, 1. $\{q_2, q_4\}$ are equivalent.
  - What about q_1 and q_3?

Table-Filling Algorithm ii

- Here is an algorithm to find all equivalent states.
  - (Basis) If $p \in F$ but $q \notin F$, then $\{p, q\}$ is distinguishable;
  - (Inductive) Let $p, q \in Q$, $a \in \Sigma$, $r = \delta(p, a)$, and $s = \delta(q, a)$. If $\{r, s\}$ is distinguishable, then $\{p, q\}$ is distinguishable.
- Proof sketch:
  - If $p \in F$ but $q \notin F$, $p = \delta(p, \epsilon) \in F$ and $q = \delta(q, \epsilon) \notin F$. $\{p, q\}$ is distinguishable.
  - By inductive hypothesis, there is a w such that $r \xrightarrow{w} r' \in F$ but $s \xrightarrow{w} s' \notin F$ (the other case is symmetric). Then $p \xrightarrow{aw} r' \in F$ and $q \xrightarrow{aw} s' \notin F$. $\{p, q\}$ is distinguishable.

Table-Filling Algorithm iii



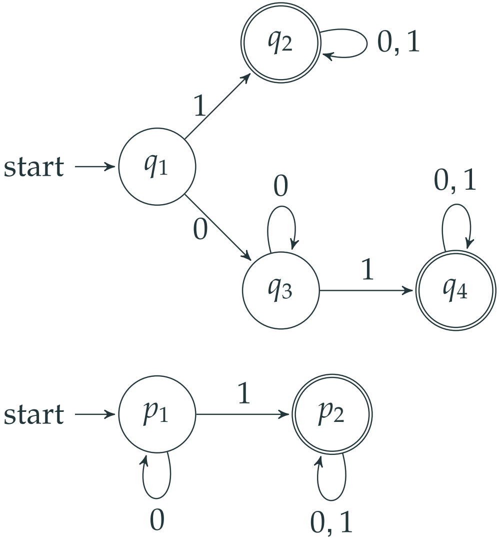
	q_1	q_2	q_3	q_4
q_1				
q_2	X			
q_3		X		
q_4	X		X	

- By the algorithm, we see $\{q_1, q_3\}$ and $\{q_2, q_4\}$ are equivalent.
- We know how to find equivalent states in a DFA.

Equivalence of DFA's i

- Now consider two DFA's M_0 and M_1.
- How do we know if $L(M_0) = L(M_1)$?
- Put M_0 and M_1 together and check if the start states are equivalent.

Equivalence of DFA's ii



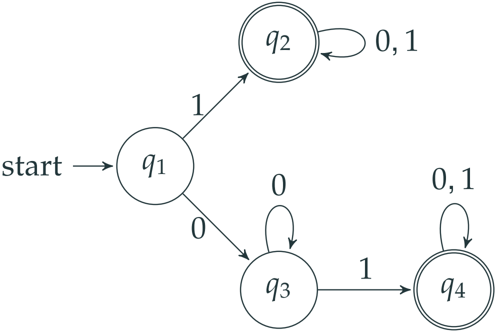
	q_1	q_2	q_3	q_4	p_1	p_2
q_1						
q_2	X					
q_3		X				
q_4	X		X			
p_1		X		X		
p_2	X		X		X	

- Since q_1 and p_1 are equivalent, both DFA's accept the same language.
- Moreover, we know $\{q_1, q_3, p_1\}$ are equivalent and $\{q_2, q_4, p_2\}$ are equivalent.

Minimization of DFA's i

- Given a DFA M, can we find a DFA M' with the minimum number of states and $L(M) = L(M')$?
- Surprisingly, the table-filling algorithm can solve the minimization problem.
- Here is the algorithm:
  - Remove all states unreachable from the initial state;
  - Use the table-filling algorithm to find equivalent states;
  - Construct M' with equivalent classes as states.

Minimization of DFA's ii



	q_1	q_2	q_3	q_4
q_1				
q_2	X			
q_3		X		
q_4	X		X	

- Equivalent classes are $E_1 = \{q_1, q_3\}$ and $E_2 = \{q_2, q_4\}$.
- $M' = (\{E_1, E_2\}, \{0, 1\}, \delta', E_1, \{E_2\})$ and

δ'	0	1
E_1	E_1	E_2
E_2	E_2	E_2

Nonregular Languages

Pumping Lemma

Lemma 15

If A is a regular language, then there is a number p such that for any $s \in A$ of length at least p, there is a partition $s = xyz$ with

1. for each $i \geq 0$, $xy^iz \in A$;
2. $|y| > 0$; and
3. $|xy| \leq p$.

Proof.

Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA recognizing A and $p = |Q|$.

Consider any string $s = s_1 s_2 \cdots s_n \in L(M)$ of length $n \geq p$. Let $r_1 = q_1, \ldots, r_{n+1}$ be the sequence of states such that $r_{i+1} = \delta(r_i, s_i)$ for $1 \leq i \leq n$. Since $n + 1 \geq p + 1 = |Q| + 1$, there are $1 \leq j < l \leq p + 1$ such that $r_j = r_l$ (why?). Choose $x = s_1 \cdots s_{j-1}$, $y = s_j \cdots s_{l-1}$, and $z = s_l \cdots s_n$. Note that $r_1 \xrightarrow{x} r_j$, $r_j \xrightarrow{y} r_l$, and $r_l \xrightarrow{z} r_{n+1} \in F$. Thus M accepts xy^iz for $i \geq 0$. Since $j \neq l$, $|y| > 0$. Finally, $|xy| \leq p$ for $l \leq p + 1$. □

Applications of Pumping Lemma

Example 16
$B = \{0^n 1^n : n \geq 0\}$ is not a regular language.

Proof.
Suppose B is regular. Let p be the pumping length given by the pumping lemma. Choose $s = 0^p 1^p$. Then $s \in B$ and $|s| \geq p$, there is a partition $s = xyz$ such that $xy^i z \in B$ for $i \geq 0$. Since $|xy| \leq p$ and $|y| > 0$, $y \in 0^+$. $xz \notin B$. A contradiction. □

Corollary 17
$C = \{w : w$ has an equal number of 0's and 1's$\}$ is not a regular language.

Proof.
Suppose C is regular. Then $B = C \cap 0^* 1^*$ is regular. □

Applications of Pumping Lemma

Example 18

$F = \{ww : w \in \{\mathtt{0}, \mathtt{1}\}^*\}$ is not a regular language.

Proof.

Suppose F is a regular language and p the pumping length. Choose $s = \mathtt{0}^p\mathtt{1}\mathtt{0}^p\mathtt{1}$. By the pumping lemma, there is a partition $s = xyz$ such that $|xy| \leq p$ and $xy^iz \in F$ for $i \geq 0$. Since $|xy| \leq p$, $y \in \mathtt{0}^+$. But then $xz \notin F$. A contradiction. □

Applications of Pumping Lemma

Example 19
$D = \{1^{n^2} : n \geq 0\}$ is not a regular language.

Proof.
Suppose D is a regular language and p the pumping length. Choose $s = 1^{p^2}$. By the pumping lemma, there is a partition $s = xyz$ such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in D$ for $i \geq 0$. Consider the strings xyz and xy^2z. We have $|xyz| = p^2$ and $|xy^2z| = p^2 + |y| \leq p^2 + p < p^2 + 2p + 1 = (p+1)^2$. Since $|y| > 0$, we have $p^2 = |xyz| < |xy^2z| < (p+1)^2$. Thus $xy^2z \notin D$. A contradiction. $\square$

Example 20
$E = \{0^i 1^j : i > j\}$ is not a regular language.

Proof.
Suppose E is a regular language and p the pumping length. Choose $s = 0^{p+1}1^p$. By the pumping lemma, there is a partition $s = xyz$ such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in E$ for $i \geq 0$. Since $|xy| \leq p$, $y \in 0^+$. But then $xz \notin E$ for $|y| > 0$. A contradiction. □

To Infinity and Beyond

ω-Automata

- We would like to generalize inputs to finite automata.
- Instead of finite input strings, let us consider an infinite input strings $\alpha = a_1 a_2 \cdots a_n \cdots$ over Σ.
- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton.
- As before, define a run $\rho = q_0 q_1 \cdots q_n \cdots$ on α to be an infinite sequence of states such that

$$\text{for all } i \geq 0, (q_i, a_{i+1}, q_{i+1}) \in \delta.$$

- What is an accepting run then?
  - Problem: there is no "final" state in an infinite run.
  - We cannot reuse the old definition.

Büchi Acceptance

- Let $\rho = q_0 q_1 \cdots q_n \cdots$ be an infinite run.
- Define
$$\text{Inf}(\rho) = \{q \in Q : q \text{ occurs infinitely many times in } \rho\}.$$
- An infinite run ρ over α on $M = (Q, \Sigma, \delta, q_0, F)$ is accepting if $\text{Inf}(\rho) \cap F \neq \emptyset$.
  - This is called Büchi acceptance
- An infinite input string α is accepted by M if there is an accepting infinite run ρ over α on M.
- Finally, define
$$L_\omega(M) = \{\alpha : \alpha \text{ is an infinite input string accepted by } M\}.$$

Example

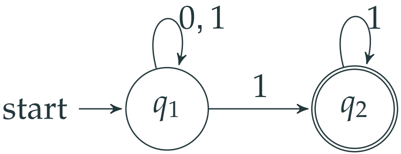


Figure 9: NFA N_6

- $L_\omega(N_6) = \{\alpha : \alpha \text{ has only finitely many 0's}\}$.
    - If there are infintiely many 0's, N_6 has to stay in q_1. It cannot pass q_2 infinitely many times.
- We will write the expression $(0+1)^*1^\omega$ to denote $L(N_6)$.

Nondeterminism

- For finite automata over finite input strings, we know nondeterminism does not give us more expressive power.
- However, nondeterministic finite automata over infinite input strings can recognize more languages than deterministic ones.

Theorem 21
$(0+1)^*1^\omega$ cannot be accepted by any deterministic finite automata.

Proof.
Suppose $D = (Q, \Sigma, \delta, q_0, F)$ is a DFA and $L(D) = (0+1)^*1^\omega$. Consider 1^ω. There is n_0 such that 1^{n_0} causes D to reach an accepting state. Now consider $1^{n_0}01^\omega$. There is n_1 such that $1^{n_0}01^{n_2}$ causes D to reach an accepting state. We can therefore construct $1^{n_0}01^{n_1}01^{n_2}0\cdots$ to cause D to pass through F infinitely many times. A contradiction. □

Remark

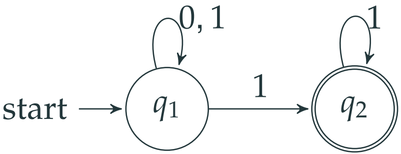


Figure 10: NFA N_6

- The proof does not work for NFA.
- Consider again the NFA N_6.
- 1 causes N_6 to reach q_2. 101 causes N_6 to reach q_2, etc. There is no problem.
- However, 101 passes q_2 only once. Similarly, 10101, 1010101, ... pass q_2 only once.
- Because N_6 is nondeterministic, infinite runs may not be the "limit" of their finite prefixes.

The Class of Regular ω-Languages

- Define
$$\mathcal{R}_\omega = \{L_\omega(M) : M \text{ is an NFA with Büchi acceptance }\}.$$
- $\mathcal{R}_\omega$ is called the class of regular ω-languages.
- Moreover, it is known the class of regular ω-language is closed under intersection, union, and complement.
- Under Büchi acceptance, nondeterminism increases the expressive power. We have
$$\{L_\omega(D) : D \text{ is a DFA with Büchi acceptance }\} \subsetneq \mathcal{R}_\omega.$$

Concluding Remarks

Where to Go

- Automata theory is a rich field.
- It is widely studied in computational complexity, formal verification, and natural language processing.
- You will see applications of automata theory in formal verification later.