

Suggested Solutions #1

[Compiled on September 5, 2017]

1. Use the semantic method to argue the validity of the following Σ_E -formulae, or identify a counterexample (a falsifying T_E -interpretation).

- (a) $f(x, y) = f(y, x) \rightarrow f(a, y) = f(y, a)$
 (b) $f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \rightarrow g(f(x)) = x$

Solution.

- (a) There is a falsifying interpretation where $f(m, n) = m^n$ for all $m, n \in \mathbb{N}$, $x = 2$, $y = 2$, and $a = 3$.
 (b) Assume there is an interpretation M such that $M \not\models f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \rightarrow g(f(x)) = x$. Then,

1. $M \models f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x$
2. $M \not\models g(f(x)) = x$
3. $M \models f(y) = x$ (by 1)
4. $M \models g(f(y)) = g(x)$ (by 3 and function congruence)
5. $M \models f(g(f(y))) = f(g(x))$ (by 4 and function congruence)
6. $M \models f(g(f(y))) = x$ (by 1)
7. $M \models f(g(x)) = f(g(f(y)))$ (by 5 and symmetry)
8. $M \models f(g(x)) = x$ (by 6, 7 and transitivity)
9. $M \models f(g(x)) = g(f(x))$ (by 1)
10. $M \models g(f(x)) = f(g(x))$ (by 9 and symmetry)
11. $M \models g(f(x)) = x$ (by 10, 8 and transitivity)

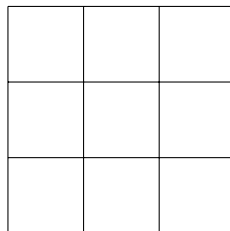
Since we find a contradiction, the formula is T_E -valid.

□

2. Given the following 3×3 grid, we would like to find a way to fill the grid with numbers from 1 to 9 such that

- summations of every row, every column, and every diagonal are the same, and
- each number can appear only once.

Try to write an SMT formula such that the way exists if the SMT formula is satisfiable.



Solution. Let $x_{i,j}$ denote the number in the cell at i -th row and j -th column. Assume that there is a sum sum . Each number can be from 1 to 9.

$$\bigwedge_{i=\{1,2,3\}, j=\{1,2,3\}} \left(\bigvee_{1 \leq k \leq 9} x_{i,j} = k \right) \quad (\text{Range})$$

Summations of every row, every column, and every diagonal are the same.

$$\begin{aligned} & (\bigwedge_{1 \leq i \leq 3} (\bigoplus_{1 \leq j \leq 3} x_{i,j} = sum)) \\ \wedge & (\bigwedge_{1 \leq j \leq 3} (\bigoplus_{1 \leq i \leq 3} x_{i,j} = sum)) \\ \wedge & (\bigwedge_{i \in \{1,3\}} (x_{1,i} + x_{2,2} + x_{3,4-i} = sum)) \end{aligned} \quad (\text{Equal})$$

Each number can appear only once.

$$\bigwedge_{0 \leq n \leq 7, n < m \leq 8} x_{n/3+1, (n \bmod 3)+1} \neq x_{m/3+1, (m \bmod 3)+1} \quad (\text{Distinct})$$

Then, we can find a solution if the SMT formula $Range \wedge Equal \wedge Distinct$ is satisfiable. \square

3. Apply the decision procedure for T_E to the following Σ_E -formulae. Provide a level of details as in slides.

- (a) $f(x, y) = f(y, x) \wedge f(a, y) \neq f(y, a)$
- (b) $f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \wedge g(f(x)) \neq x$
- (c) $f(f(f(a))) = f(f(a)) \wedge f(f(f(f(a)))) = a \wedge f(a) \neq a$
- (d) $p(x) \wedge f(f(x)) = x \wedge f(f(f(x))) = x \wedge \neg p(f(x))$

Solution.

(a)

$$\begin{aligned} & \{\{a\}, \{x\}, \{y\}, \{f(x, y)\}, \{f(y, x)\}, \{f(a, y)\}, \{f(y, a)\}\} \\ & \{\{a\}, \{x\}, \{y\}, \{f(x, y), f(y, x)\}, \{f(a, y)\}, \{f(y, a)\}\} \quad (f(x, y) = f(y, x)) \end{aligned}$$

T_E -satisfiable

(b)

$$\begin{aligned} & \{\{x\}, \{y\}, \{f(x)\}, \{g(x)\}, \{f(y)\}, \{f(g(x))\}, \{g(f(x))\}, \{g(f(y))\}, \{f(g(f(y)))\}\} \\ & \quad (f(g(x)) = g(f(x))) \\ & \{\{x\}, \{y\}, \{f(x)\}, \{g(x)\}, \{f(y)\}, \{f(g(x)), g(f(x))\}, \{g(f(y))\}, \{f(g(f(y)))\}\} \\ & \quad (f(g(f(y))) = x) \\ & \{\{x, f(g(f(y)))\}, \{y\}, \{f(x)\}, \{g(x)\}, \{f(y)\}, \{f(g(x)), g(f(x))\}, \{g(f(y))\}\} \\ & \quad (f(y) = x) \\ & \{\{x, f(g(f(y))), f(y)\}, \{y\}, \{f(x)\}, \{g(x)\}, \{f(g(x)), g(f(x))\}, \{g(f(y))\}\} \\ & \quad (\text{function congruence}) \\ & \{\{x, f(g(f(y))), f(y), f(g(x)), g(f(x))\}, \{y\}, \{f(x)\}, \{g(x), g(f(y))\}\} \\ & \quad (\text{function congruence}) \\ & \{\{x, f(g(f(y))), f(y), f(g(x)), g(f(x)), g(x), g(f(y))\}, \{y\}, \{f(x)\}\} \end{aligned}$$

T_E -unsatisfiable

(c)

$$\begin{array}{ll}
\{\{a\}, \{f(a)\}, \{f(f(a))\}, \{f(f(f(a)))\}, \{f(f(f(f(a))))\}\} & \\
\{\{a\}, \{f(a)\}, \{f(f(a)), f(f(f(a)))\}, \{f(f(f(f(a))))\}\} & (f(f(f(a))) = f(f(a))) \\
\{\{a\}, \{f(a)\}, \{f(f(a)), f(f(f(a))), f(f(f(f(a))))\}\} & (\text{function congruence}) \\
\{\{a, f(f(a)), f(f(f(a))), f(f(f(f(a))))\}, \{f(a)\}\} & (f(f(f(f(a)))) = a) \\
\{\{a, f(f(a)), f(f(f(a))), f(f(f(f(a))))\}, f(a)\} & (\text{function congruence})
\end{array}$$

T_E -unsatisfiable

(d) Consider the formula $f_p(x) = \bullet \wedge f(f(x)) = x \wedge f(f(f(x))) = x \wedge f_p(f(x)) \neq \bullet$ instead.

$$\begin{array}{ll}
\{\{\bullet\}, \{x\}, \{f(x)\}, \{f_p(x)\}, \{f(f(x))\}, \{f_p(f(x))\}, \{f(f(f(x)))\}\} & \\
\{\{\bullet, f_p(x)\}, \{x\}, \{f(x)\}, \{f(f(x))\}, \{f_p(f(x))\}, \{f(f(f(x)))\}\} & (f_p(x) = \bullet) \\
\{\{\bullet, f_p(x)\}, \{x, f(f(x))\}, \{f(x)\}, \{f_p(f(x))\}, \{f(f(f(x)))\}\} & (f(f(x)) = x) \\
\{\{\bullet, f_p(x)\}, \{x, f(f(x))\}, \{f(x), f(f(f(x)))\}, \{f_p(f(x))\}\} & (\text{function congruence}) \\
\{\{\bullet, f_p(x)\}, \{x, f(f(x)), f(x), f(f(f(x)))\}, \{f_p(f(x))\}\} & (f(f(f(x))) = x) \\
\{\{\bullet, f_p(x), x, f(f(x)), f(x), f(f(f(x))), f_p(f(x))\}\} & (\text{function congruence})
\end{array}$$

T_E -unsatisfiable

□

4. Apply the decision procedure for T_{cons} to the following T_{cons} -formulae. Please write down the call sequence to the MERGE procedure, draw the final DAG, and draw the final DAG.

(a) $car(x) = y \wedge cdr(x) = z \wedge x \neq cons(y, z)$

(b) $\neg atom(x) \wedge car(x) = y \wedge cdr(x) = z \wedge x \neq cons(y, z)$

Solution.

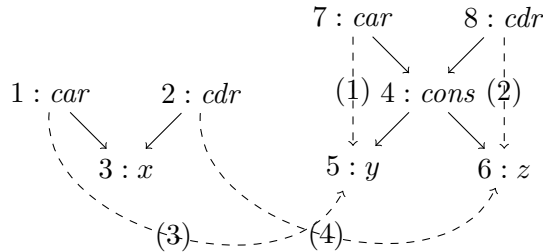
(a) The following is the initial DAG.



The following is the merge sequences.

- (1) Add node 7 : $car(cons(y, z))$ and MERGE 7 5 (by left projection)
- (2) Add node 8 : $cdr(cons(y, z))$ and MERGE 8 6 (by right projection)
- (3) MERGE 1 5 (by $car(x) = y$)
- (4) MERGE 2 6 (by $cdr(x) = z$)

The following is the final DAG.

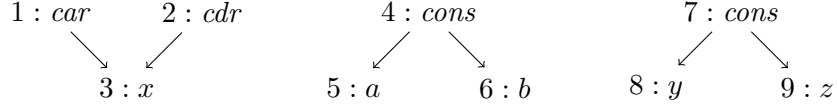


Consider $x \neq \text{cons}(y, z)$, we have $\text{FIND } 3 \neq \text{FIND } 4$. Thus, the formula is T_{cons} -satisfiable.

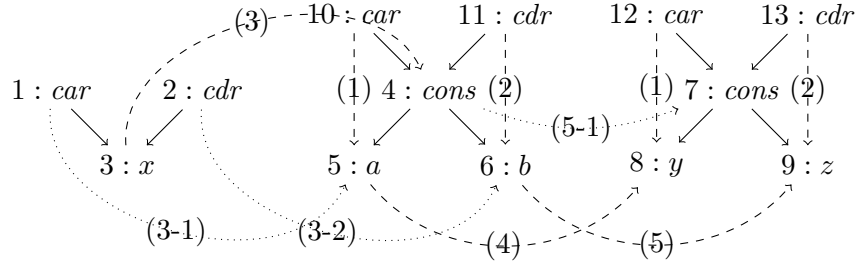
(b) Preprocess the formula and get the following one:

$$x = \text{cons}(a, b) \wedge \text{car}(x) = y \wedge \text{cdr}(x) = z \wedge x \neq \text{cons}(y, z).$$

Below is the initial DAG.



- (1) Add nodes 10 : $\text{car}(\text{cons}(a, b))$ and 12 : $\text{car}(\text{cons}(y, z))$, and MERGE 10 5 and MERGE 12 8 (by left projection)
- (2) Add nodes 11 : $\text{cdr}(\text{cons}(a, b))$ and 13 : $\text{cdr}(\text{cons}(y, z))$, and MERGE 11 6 and MERGE 13 9 (by right projection)
- (3) MERGE 3 4 (by $x = \text{cons}(a, b)$)
 - (3-1) MERGE 1 10 (by function congruence)
 - (3-2) MERGE 2 11 (by function congruence)
- (4) MERGE 1 8 (by $\text{car}(x) = y$)
- (5) MERGE 2 9 (by $\text{cdr}(x) = z$)
 - (5-1) MERGE 4 7 (by function congruence)



Consider $x \neq \text{cons}(y, z)$, we have $\text{FIND } 3 = \text{FIND } 7 = 7$. Thus, the formula is T_{cons} -unsatisfiable. □

5. Apply the decision procedure for quantifier-free T_A to the following Σ_A -formulae.

- (a) $a\langle i \triangleleft e \rangle[j] = e \wedge i \neq j$
- (b) $a\langle i \triangleleft e \rangle\langle j \triangleleft f \rangle[k] = g \wedge j \neq k \wedge i = j \wedge a[k] \neq g$

Solution.

(a) Consider the following two cases.

- Case 1: $i = j$. The formula becomes

$$i = j \wedge e = e \wedge i \neq j$$

which is T_E -unsatisfiable.

- Case 2: $i \neq j$. The formula becomes

$$i \neq j \wedge a[j] = e \wedge i \neq j$$

which is T_A -satisfiable because the following formula

$$i \neq j \wedge f_a(j) = e \wedge i \neq j$$

is T_E -satisfiable.

Conclusion: T_A -satisfiable.

- (b) Consider the following cases where the conversion from T_A formulas (without writing operations) to T_E formulas is applied by not shown here.

- Case 1: $j = k$. The formula becomes

$$j = k \wedge f = g \wedge j \neq k \wedge i = j \wedge a[k] \neq g$$

which is T_A -unsatisfiable.

- Case 1: $j \neq k$. The formula becomes

$$j \neq k \wedge a\langle i \triangleleft e \rangle[k] = g \wedge j \neq k \wedge i = j \wedge a[k] \neq g.$$

We have two sub-cases.

- Case 1(a): $i = k$. The formula becomes

$$i = k \wedge j \neq k \wedge e = g \wedge j \neq k \wedge i = j \wedge a[k] \neq g$$

which is T_A -unsatisfiable.

- Case 1(b): $i \neq k$. The formula becomes

$$i \neq k \wedge j \neq k \wedge a[k] = g \wedge j \neq k \wedge i = j \wedge a[k] \neq g$$

which is T_A -unsatisfiable.

Conclusion: T_A -unsatisfiable.

□