Programming
Research
Laboratory

FLOLAC '16

Elementary logic

Intuitionistic propositional logic and natural deduction

7 July 2016

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An elementary theorem and its proof

Theorem. If a function is bijective, it has a two-sided inverse.

Proof. Suppose that $f : A \to B$ is bijective. Surjectivity of f implies that there is a function $g : B \to A$ such that $f\,(g\ b) = b$ for all $b : B$. It remains to show that $g\,(f\ a) = a$ for all $a : A$, which holds because

$$
\begin{array}{ll}
 & g\,(f\ a) = a \\
\Leftarrow & \quad \{\, f \text{ injective} \,\} \\
 & f\,(g\,(f\ a)) = f\ a \\
\Leftarrow & \quad \{\, \text{substituting } f\ a \text{ for } b \,\} \\
 & f\,(g\ b) = b \quad \forall b : B
\end{array}
$$

□

How to prove an elementary theorem

Translate (desugar) a statement in a natural language into a proposition in a logical language. For example:

$$
\begin{array}{l}
\forall f : A \to B. \\
\quad (\forall a : A.\ \forall a' : A.\ f\ a = f\ a' \Rightarrow a = a') \wedge (\forall b : B.\ \exists a : A.\ f\ a = b) \\
\qquad \Rightarrow \exists g : B \to A. \\
\qquad\qquad (\forall b : B.\ f\ (g\ b) = b) \wedge (\forall a : A.\ g\ (f\ a) = a)
\end{array}
$$

Then follow the rules of logic to prove the proposition. For example, to prove

- "$\forall x : A.\ P$": Suppose that x is an arbitrary element of A. Continue to prove that P holds for x.
- "$\exists x : A.\ P$": Find an element of A and prove that P holds for that element.
- "$P \Rightarrow Q$": Suppose that P is true. Continue to prove Q.
- "$P \wedge Q$": Give two proofs respectively proving P and Q.
- ...

Formal logic...

... studies the general forms of propositions and valid inferences, ...

- For an extreme example, the truth of the following proposition is determined by the way we use the connectives alone:

  if *herba viridi* **and** *area est infectum*, **then** *area est infectum*

  The actual meanings/structures of the two propositions "*herba viridi*" and "*area est infectum*" do not matter.

... and emphasises *symbolisation*.

- Concise and unambiguous
- Amenable to mathematical treatment
- Mechanisable

Propositional logic

For this course, we focus on the *propositional connectives* (constants): conjunction ($\wedge$), disjunction ($\vee$), implication ($\to$), and falsity ($\perp$).

Formally, define a set of *formulas* representing generic propositions built up from the connectives.

Definition. Given a set $\mathcal{PV}$ of variable names, the set $\textsc{Prop}$ of propositional formulas is *inductively* defined by the following rules:

- $\mathcal{PV} \subseteq \textsc{Prop}$;
- $\perp \in \textsc{Prop}$;
- if φ, $\psi \in \textsc{Prop}$, then $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \to \psi \in \textsc{Prop}$.

Think of this as a datatype definition in a functional language:

$$\begin{aligned} \textbf{type}\ \mathit{PV} &= \mathit{String} \\ \textbf{data}\ \mathit{Prop} &= \mathit{Var}\ \mathit{PV} \mid \mathit{Falsity} \\ &\mid\ \mathit{Conj}\ \mathit{Prop}\ \mathit{Prop} \mid \mathit{Disj}\ \mathit{Prop}\ \mathit{Prop} \mid \mathit{Impl}\ \mathit{Prop}\ \mathit{Prop} \end{aligned}$$

Intuitionistic meaning of proposition

A proposition (à la Heyting) expresses an intention/expectation of a *proof*. A proposition is said to be true if an intended/expected proof exists.

To explain the meaning of a proposition, we describe what counts as a proof of that proposition.

- A proof of $\varphi \wedge \psi$ is a proof of φ and a proof of ψ.
- A proof of $\varphi \vee \psi$ is either a proof of ψ or a proof of ψ.
- A proof of $\varphi \to \psi$ is a way of constructing a proof of ψ given a proof of φ.
- There is no proof of $\perp$.

Formalising propositional deduction: judgement

When constructing a proof, we shift from one state of mind to another, keeping track of what are assumed to be true and what is left to be proved. This "state of mind" can be formalised as a *judgement*

$$\Gamma \vdash \varphi$$

where Γ is a list of propositions assumed to be true, and φ is a proposition we wish to prove.

Example. $(\mathtt{A} \land \mathtt{B}) \to (\mathtt{B} \land \mathtt{A})$ is a proposition, and we write the judgement $\vdash (\mathtt{A} \land \mathtt{B}) \to (\mathtt{B} \land \mathtt{A})$ to say that we expect the proposition to be true without assuming anything.

Formalising propositional inference: inference rule

A valid shift from one state of mind to another is formalised as an *inference rule*

$$\frac{J_0 \qquad \cdots \qquad J_{n-1}}{J}$$

relating the judgements above the line, called the *premises*, and the judgement below the line, called the *conclusion*, such that when the premises are correct, the conclusion is also correct.

Example. There is an inference rule

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\ (\to\mathsf{I})$$

which can be instantiated to

$$\frac{\mathtt{A} \wedge \mathtt{B} \vdash \mathtt{B} \wedge \mathtt{A}}{\vdash (\mathtt{A} \wedge \mathtt{B}) \to (\mathtt{B} \wedge \mathtt{A})}\ (\to\mathsf{I})$$

That is, if we wish to deduce the truth of $(\mathtt{A} \wedge \mathtt{B}) \to (\mathtt{B} \wedge \mathtt{A})$, it suffices to deduce the truth of $\mathtt{B} \wedge \mathtt{A}$ supposing the truth of $\mathtt{A} \wedge \mathtt{B}$.

Formalising propositional inference: derivation

Starting from a judgement, we apply inference rules until no premises are left. The resulting tree of the applications of inference rules is called a *derivation* of the judgement.

Example. Below is a derivation that explains why the proposition $(\mathtt{A} \wedge \mathtt{B}) \to (\mathtt{B} \wedge \mathtt{A})$ is true:

$$\dfrac{\dfrac{\dfrac{}{\mathtt{A} \wedge \mathtt{B} \vdash \mathtt{A} \wedge \mathtt{B}}\,(\text{assum})}{\mathtt{A} \wedge \mathtt{B} \vdash \mathtt{B}}\,(\wedge\text{ER}) \qquad \dfrac{\dfrac{}{\mathtt{A} \wedge \mathtt{B} \vdash \mathtt{A} \wedge \mathtt{B}}\,(\text{assum})}{\mathtt{A} \wedge \mathtt{B} \vdash \mathtt{A}}\,(\wedge\text{EL})}{\dfrac{\mathtt{A} \wedge \mathtt{B} \vdash \mathtt{B} \wedge \mathtt{A}}{\vdash (\mathtt{A} \wedge \mathtt{B}) \to (\mathtt{B} \wedge \mathtt{A})}\,(\to\text{I})}\,(\wedge\text{I})$$

Definition. A proposition φ is called a *theorem* exactly when $\vdash \varphi$ is derivable.

Natural deduction

Invented by Gentzen, natural deduction is a collection of inference rules intended to capture the natural way in which mathematicians construct proofs. The intuitionistic variant we are introducing is named NJ by Gentzen.

The simplest inference rule is the assumption rule:

$$\frac{}{\Gamma \vdash \varphi}\ (\text{assum})$$

with the *side condition* that φ appears in Γ.

For each propositional connective (constant), there are

- zero or more *introduction* rules saying how to build a proof involving the connective, and
- zero or more *elimination* rules saying how to use a proof involving the connective.

Conjunction

- Introduction:

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \land \psi}\ (\land\mathsf{I})$$

- Elimination:

$$\frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \varphi}\ (\land\mathsf{EL}) \qquad \frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \psi}\ (\land\mathsf{ER})$$

Exercise. Derive

$$(\mathtt{A} \land \mathtt{B}) \land \mathtt{C} \vdash \mathtt{A} \land (\mathtt{B} \land \mathtt{C})$$

Implication

- Introduction:

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\ (\to\mathsf{I})$$

- Elimination:

$$\frac{\Gamma \vdash \varphi \to \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}\ (\to\mathsf{E})$$

Exercise. Derive

$$\vdash (\mathtt{A} \to (\mathtt{B} \land \mathtt{C})) \to ((\mathtt{A} \to \mathtt{B}) \land (\mathtt{A} \to \mathtt{C}))$$

We will use $\varphi \leftrightarrow \psi$ as a shorthand for $(\varphi \to \psi) \land (\psi \to \varphi)$.

Disjunction

- Introduction:

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ (\vee\text{IL}) \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ (\vee\text{IR})$$

- Elimination:

$$\frac{\Gamma \vdash \varphi \vee \psi \qquad \Gamma, \varphi \vdash \vartheta \qquad \Gamma, \psi \vdash \vartheta}{\Gamma \vdash \vartheta}\ (\vee\text{E})$$

Exercise. Derive

$$\vdash (\mathtt{A} \vee \mathtt{B}) \rightarrow (\mathtt{B} \vee \mathtt{A})$$

Falsity

- Introduction: none
- Elimination:

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\,(\bot\mathsf{E})$$

We use $\neg\varphi$ as a shorthand for $\varphi \to \bot$ and $\top$ for $\bot \to \bot$.

Exercise. Derive

$$\vdash ((\mathtt{A} \vee \mathtt{B}) \wedge \neg\mathtt{B}) \to \mathtt{A}$$

Non-provable propositions

We can prove	but not
$\neg\neg(\mathtt{A} \vee \neg\mathtt{A})$	$\mathtt{A} \vee \neg\mathtt{A}$ (*law of excluded middle*)
$A \to \neg\neg\mathtt{A}$	$\neg\neg\mathtt{A} \to \mathtt{A}$ (*principle of indirect proof*)
$(\neg\mathtt{A} \vee \neg\mathtt{B}) \to \neg(\mathtt{A} \wedge \mathtt{B})$	$\neg(\mathtt{A} \wedge \mathtt{B}) \to (\neg\mathtt{A} \vee \neg\mathtt{B})$
$(\mathtt{A} \to \mathtt{B}) \to (\neg\mathtt{B} \to \neg\mathtt{A})$	$(\neg\mathtt{B} \to \neg\mathtt{A}) \to (\mathtt{A} \to \mathtt{B})$

Intuitionistic vs classical logic

We can obtain a natural deduction system NK for classical logic by adding to NJ an inference rule encoding the law of excluded middle or the principle of indirect proof (double negation elimination):

$$\frac{}{\Gamma \vdash \varphi \vee \neg\varphi}\ \text{(LEM)} \qquad \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}\ (\neg\neg\text{E})$$

Intuitionistic logic is more precise about constructivity: In intuitionistic logic, A and $\neg\neg A$ have different meanings, but in classical logic they are indistinguishable.

Theorem (Glivenko)**.** $\Gamma \vdash_{\mathrm{NK}} \varphi$ if and only if $\neg\neg\Gamma \vdash_{\mathrm{NJ}} \neg\neg\varphi$.

Proof sketch. ($\Leftarrow$) Every proposition in $\neg\neg\Gamma$ is implied by the corresponding one in Γ, so $\Gamma \vdash_{\mathrm{NK}} \neg\neg\varphi$ is derivable, which implies that $\Gamma \vdash_{\mathrm{NK}} \varphi$ is derivable by double negation elimination.

($\Rightarrow$) By induction on NK derivations. $\square$