Exercise

Simplex Method

Is the following $T_{\mathbb{Q}}$-formula satisfiable?

$$\begin{array}{rcl} x+y+z & \geq & 1 \\ x-y+z & \geq & 2 \\ 2x+y-2z & \leq & 5 \end{array}$$

Use the simplex method to find a solution with Z3, and verify your solution in Z3.

Solution:

We first introduce additional variables $x^+, x^-, y^+, y^-, z^+, z^-$ s.t. $x = x^+ - x^-$, $y = y^+ - y^-$, $z = z^+ - z^-$, and all these extra variables are non-negative.

For the two $\geq$ constraints, we have to introduce two variables a_1, a_2 to transform them to $\leq$ constraints.

Let $\overline{x} := [x^+ \ x^- \ y^+ \ y^- \ z^+ \ z^-]^T$ and $\overline{z} = [a_1 \ a_2]^T$. The problem thereafter becomes:

$$\begin{array}{rcl} -\overline{x} & \leq & \overline{0} \\ D_1 \, \overline{x} & \leq & \overline{g_1} = \begin{bmatrix} 5 \end{bmatrix} \\ D_2 \, \overline{x} - \overline{z} & \leq & \overline{g_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array}$$

With

$$\begin{array}{rcl} D_1 & = & \begin{bmatrix} 2 & -2 & 1 & -1 & -2 & 2 \end{bmatrix} \\ D_2 & = & \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix} \end{array}$$

Also we add the goal that we want to maximize:

$$\begin{array}{rl} & \overline{1}^T (D_2 \, \overline{x} - \overline{z}) \\ = & \begin{bmatrix} 1 & 1 \end{bmatrix} \left(\begin{bmatrix} x^+ - x^- + y^+ - y^- + z^+ - z^- \\ x^+ - x^- - y^+ + y^- + z^+ - z^- \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) \\ = & \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x^+ - x^- + y^+ - y^- + z^+ - z^- - a_1 \\ x^+ - x^- - y^+ + y^- + z^+ - z^- - a_2 \end{bmatrix} \\ = & \begin{bmatrix} 2x^+ - 2x^- + 2z^+ - 2z^- - a_1 - a_2 \end{bmatrix} \\ = & \begin{bmatrix} 2 & -2 & 0 & 0 & 2 & -2 & -1 & -1 \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{z} \end{bmatrix} \end{array}$$

Let $\overline{v} = \begin{bmatrix} \overline{x} \\ \overline{z} \end{bmatrix}$. The corresponding M_0 for the problem is:

max:

$$\overline{c}^T \overline{v} = \begin{bmatrix} 2 & -2 & 0 & 0 & 2 & -2 & -1 & -1 \end{bmatrix} \overline{v}$$

subject to:

$$A\overline{v} = \begin{bmatrix} & & & -\mathbb{I}_8 & & & & \\ 2 & -2 & 1 & -1 & -2 & -2 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & -1 \end{bmatrix} \overline{v} \leq \begin{bmatrix} \overline{0} \\ 5 \\ 1 \\ 2 \end{bmatrix} = \overline{b}$$

Iteration 1

We find the initial vertex $\overline{v_0} = \overline{0}$ since we simply pick the first 8 rows as the defining constraint $A_0\overline{v_0} = b_0$. Then, we check if $\overline{v_0}$ attains maximum by first solving $A_0\overline{u_0} = \overline{c}$.

$$A_0\overline{u_0} = \overline{c} \Leftrightarrow -\mathbb{I}_8\overline{u_0} = \overline{c} \Leftrightarrow \overline{u_0} = -\overline{c} \Leftrightarrow \overline{u_0} = \begin{bmatrix} -2 & 2 & 0 & 0 & -2 & 2 & 1 & 1 \end{bmatrix}^T$$

$\because \overline{u_0} \ngeq \overline{0}$, $\overline{v_0}$ is not optimal. We find row 1 in $\overline{u_0}$ is -2; therefore we solve $A_0^T\overline{y_0} = -e_1$ to find direction $\overline{y_0}$.

$$A_0^T\overline{y_0} = -e_1 \Leftrightarrow -\mathbb{I}_8^T\overline{y_0} = -e_1 \Leftrightarrow \overline{y_0} = e_1 \Leftrightarrow \overline{y_0} = \begin{bmatrix} 1 \\ \overline{0} \end{bmatrix}$$

Then, computing $A\overline{y_0} \Rightarrow A\overline{y_0} = \begin{bmatrix} -1 & 0 \ldots 0 & 2 & 1 & 1 \end{bmatrix}^T \Rightarrow A\overline{y_0} \nleq \overline{0} \Rightarrow \lambda_0$ exists.

Here, we want to find maximum λ_0 that satisfies $A(\overline{v_0} + \lambda_0\overline{y_0}) \leq \overline{b}$. We however only have to consider the rows with positive value in $A\overline{y_0}$. In this case, we only have to consider row 9, 10, and 11.

$$\begin{bmatrix} 2 & -2 & 1 & -1 & -2 & -2 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & -1 \end{bmatrix} (\overline{v_0} + \lambda_0\overline{y_0}) \leq \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad \lambda_0 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \text{maximum } \lambda_0 = 1 \text{ bounded by row 10 of A}$$

Iteration 2

Therefore, $\overline{v_1} = \overline{v_0} + \lambda_0\overline{y_0} = \overline{0} + \begin{bmatrix} 1 \\ \overline{0} \end{bmatrix} = \begin{bmatrix} 1 \\ \overline{0} \end{bmatrix}$. And we use row 10 in $A, \overline{b}$ to replace row 1 in $A_0, \overline{b_0}$ to build $A_1, \overline{b1}$. The defining constraint becomes

$$A_1\overline{v_1} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & -1 & 0 \\ \overline{0} & & & & -\mathbb{I}_7 & & & \end{bmatrix} \overline{v_1} = \begin{bmatrix} 1 \\ \overline{0} \end{bmatrix} = \overline{b_1}$$

Solving $A_1\overline{u_1} = \bar{c}$, we obtain $\overline{u_1} = \begin{bmatrix}2 & 0 & 2 & -2 & 0 & 0 & -1 & 1\end{bmatrix}^T$.

$\because \overline{u_1} \not\geq \bar{0}$, $\overline{v_1}$ is not optimal.

Find row 4 in $\overline{u_0}$ is -2.

Solve $A_1^T\overline{y_1} = -e_4 \Rightarrow \overline{y_1} = \begin{bmatrix}1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\end{bmatrix}$.

Computing $A\overline{y_1} = \begin{bmatrix}-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 2\end{bmatrix}^T \Rightarrow A\overline{y_1} \not\leq \bar{0} \Rightarrow \lambda_1$ exists.

Consider only row 9 and 11.

$$\begin{bmatrix}2 & -2 & 1 & -1 & -2 & -2 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & -1\end{bmatrix}(\overline{v_1} + \lambda_1\overline{y_1}) \leq \begin{bmatrix}5 \\ 2\end{bmatrix}$$

$$\Rightarrow \quad \lambda_1 \begin{bmatrix}1 \\ 2\end{bmatrix} \leq \begin{bmatrix}4 \\ 1\end{bmatrix} \Rightarrow \text{maximum } \lambda_1 = \tfrac{1}{2} \text{ bounded by row 11 of A}$$

Iteration 3

Therefore, $\overline{v_2} = \overline{v_1} + \lambda_0\overline{y_1} = \begin{bmatrix}\frac{3}{2} & 0 & 0 & \frac{1}{2} & 0\ldots0\end{bmatrix}^T$. And we use row 11 in $A, \bar{b}$ to replace row 4 in $A_1, \overline{b_1}$ to build $A_2, \overline{b2}$. The defining constraint becomes

$$A_2\overline{v_2} = \begin{bmatrix}1 & -1 & 1 & -1 & 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & -1 \\ & & \mathbb{0}_4 & & & & -\mathbb{I}_4 & \end{bmatrix}\overline{v_2} = \begin{bmatrix}1 \\ 0 \\ 0 \\ 2 \\ \bar{0}\end{bmatrix} = \overline{b_2}$$

Solving $A_2\overline{u_2} = \bar{c}$, we obtain $\overline{u_2} = \begin{bmatrix}1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\end{bmatrix}^T$.

$\because \overline{u_2} \geq \bar{0}$, $\overline{v_2}$ is optimal.

We now have to check if $\bar{c}^T\overline{v_2} = \bar{1}^T\overline{g_2}$ to say G is satisfiable.

$$\begin{aligned}\bar{c}^T\overline{v_2} &= \begin{bmatrix}2 & -2 & 0 & 0 & 2 & -2 & -1 & -1\end{bmatrix}\begin{bmatrix}\frac{3}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0\end{bmatrix}^T \\ &= 3 \\ &= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 \\ 2\end{bmatrix} = \bar{1}^T\overline{g_2}\end{aligned}$$

Hence, G is satisfiable.

To obtain satisfiable instance for original problem:

$$\begin{aligned}&\overline{v_2} = \begin{bmatrix}\frac{3}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0\end{bmatrix}^T \\ \Rightarrow\quad & x^+ = \tfrac{3}{2}, x^- = 0, y^+ = 0, y^- = \tfrac{1}{2}, z^+ = 0, z^- = 0 \\ \Rightarrow\quad & x = \tfrac{3}{2}, y = -\tfrac{1}{2}, z = 0\end{aligned}$$