First-order Logic (Session 1)

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Limitations of propositional logic

Consider the following classical argument:

(1) All men are mortal
(2) Socrates is a man

Therefore: Socrates is mortal

Can you express this in propositional logic?

Limitations of propositional logic

Here is an attempt:

(1) All men are mortal:

Man(Socrates) -> Mortal(Socrates)
Man(Anthony) -> Mortal(Anthony)
Man(謝橋) -> Mortal(謝橋)

...

(2) Socrates is a man:

Man(Socrates)

Therefore: Socrates is mortal

Mortal(Socrates)

Problem:

How big is this formula?

A better solution

Extend the logic to easily refer to "all men"

$$\forall x : \text{Man}(x) \rightarrow \text{Mortal}(x)$$

quantifier

predicate

Read (verbose): "For all x, if **x** is a man, then **x** is mortal"

Note: Propositions are now "predicates" which depend on **x**

Observation: two lines vs. billions of lines

Goals today

* Be familiar with basic concepts on FOL:
    1. Syntax and semantics
    2. Satisfiability, validity, and equivalence

What else can you say in FOL?

* There is a man who is not married

$$\exists x : \text{man}(x) \land \neg\text{married}(x)$$

* Every person has a mother

$$\forall x : \text{person}(x) \rightarrow (\exists y : \text{mother-of}(y, x))$$

* Some person have two mobile phones

$$\exists x \exists y \exists z : \text{person}(x) \land \text{mp}(y, x) \land \text{mp}(z, x) \land z \neq y$$

So, is it true that ...?

Q: ... FOL is just PL with quantifiers and more complex "propositions"?

A: Yes, pretty much. But this is much much more complex in fact!

Ponderables

(i) Quantifiers "quantify" over what?

(ii) Which of the following sentence are "true"?

$$\exists x : \text{man}(x) \wedge \neg \text{married}(x)$$

$$(\exists x : \text{man}(x)) \rightarrow (\forall y : \text{man}(x))$$

$$(\forall x : \text{man}(x)) \rightarrow (\exists y : \text{man}(x))$$

$$\forall x : \text{man}(x) \rightarrow \text{man}(x)$$

First-order Logic (FOL) Syntax

"Atoms" (Simplified)

(generalised propositions?)

Examples of "**atomic formulas**" ("**atoms**") in FOL:

$\mathrm{man}(x) \qquad \mathrm{even}(1) \qquad \mathrm{mp}(y, x)$

Variable

Constant

Predicate /relation

Relations have **arities** (# arguments):

- **man** and **even** have arity 1
- **mp** has arity 2

Relation with arity 0 is a proposition, eg, man("John")

"Atoms" (Simplified)

Variables: x, y, ...

Constants: 0, 1, "Anthony", "謝橋"...

Terms: variables/constants

Relation symbols (with arities): man/1, mp/2, ...

Special relation: =/2

Defn: If R/i is a relation symbol with arity i and each of $t_1, t_2, \ldots, t_i$ is a term, then

$$R(t_1, t_2, \ldots, t_i)$$

is an atomic formula.

"Formulas"

As in boolean logic, build formulas from propositions with:

$$\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus$$

In addition, formulas can be "quantified":

If F is a formula and **x** is a variable, then

$\forall x : F$ is a formula

$\exists x : F$ is a formula

Exercises

How do you build the following formulas?

$$\exists x : \mathrm{man}(x) \wedge \neg \mathrm{married}(x)$$

$$(\exists x : \mathrm{man}(x)) \rightarrow (\forall y : \mathrm{man}(x))$$

$$\forall x : \mathrm{man}(x) \rightarrow \mathrm{man}(x)$$

More exercise

* Give a definition of FOL formulas by induction/grammar

Warning

* $(\forall x{:}\ R(y))$ is an FOL formula

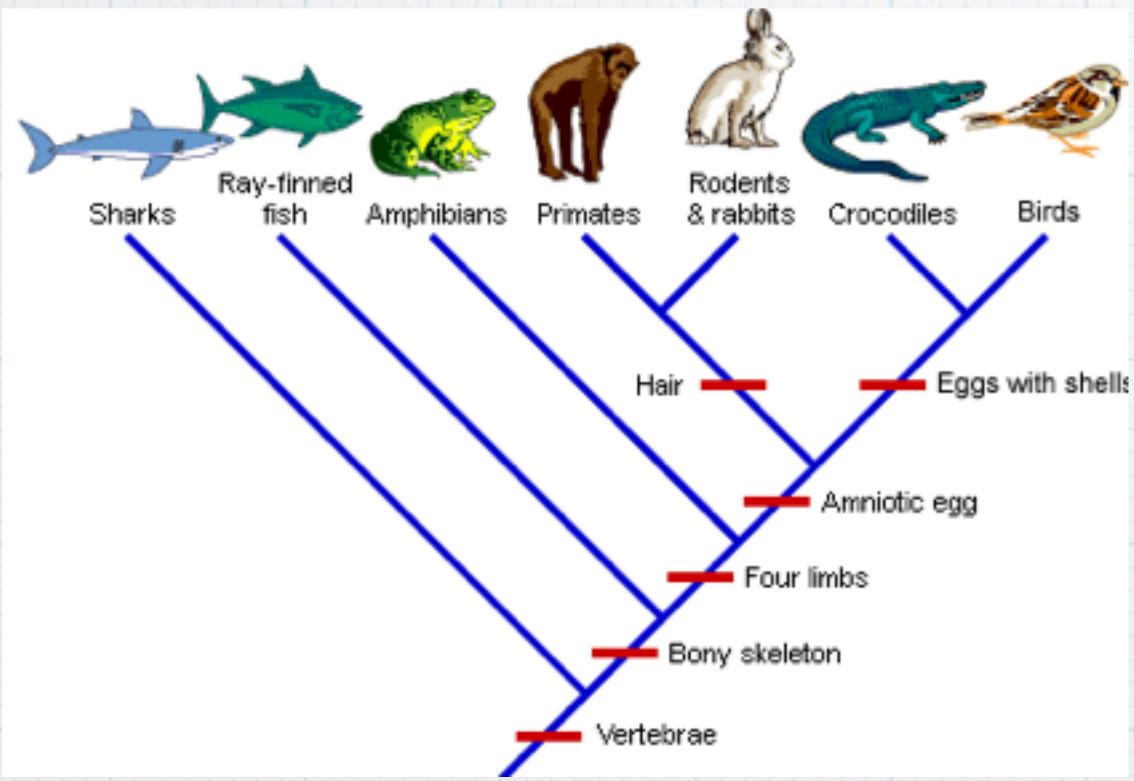
Semantics of FOL

Interpretations

What do the quantifiers quantify over?

* **Domains** **D** (a.k.a. universe)
* An **assignment** function **I** mapping:
    * Each constant **c** to an element in **D**
    * Each variable **x** to an element in D
    * Each relation symbol **R/i** to a **i**-ary relation over **D**

Example: Phylogeny tree



Relation symbols: </2, extant/1, extinct/1

Assignment:
D = {species}
I: extant/1 -> {extant species}
extinct/1 -> {extinct species}
</2 -> { (x,y) : x is an ancestor of y }

Convention: x is an ancestor of x

Example: Integer Linear Arithmetic (N,+)

Constants: 0, 1, ...

Relation symbol: **Plus/3**

Assignment:
D = {integers}
I: 0->0, 1->1, ...
I: Plus/3 -> { (x,y,z) : x+y = z }

Note: **Plus** and **+** are often confused

Truth depends on interpretations

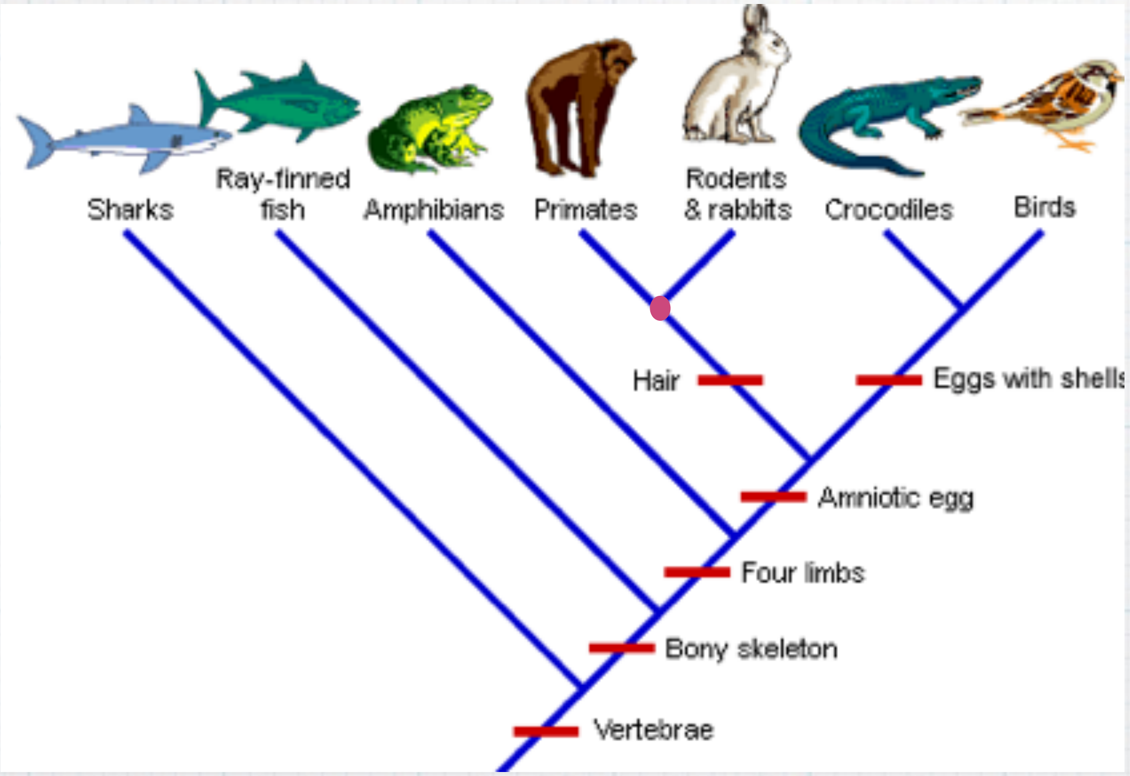
The truth/falsehood of an FOL formula depends on interpretations (just as in PL).

Need to define whether **F** is true in **I** ($I \models F$, or **I(F) = 1**) by induction on **F**:

* Atom: **I(R(x,y)) = 1** iff **(I(x),I(y))** is in **I(R)**
* AND: **I(F ∧ F') = 1** iff **I(F) =1** and **I(F') = 1**
* OR, NOT, ...: SAME

We'll deal with quantifiers later

Example 1



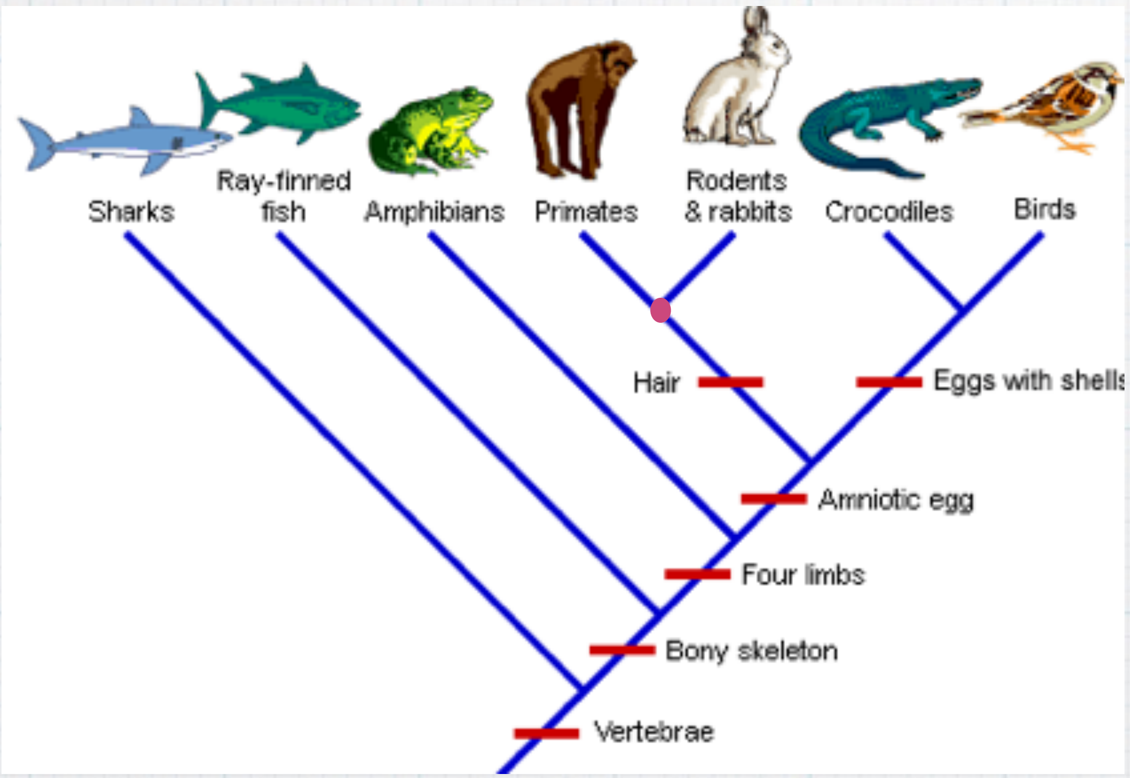
F: $z < x \wedge z < y$

Interpretation:

- x = "Primates"
- y = "Rodent"
- z = •

Is **F** true in this interpretation?

Example 2



F: z < x ∧ z < y

Interpretation:

- x = "Primates"
- y = "Rodent"
- z = "Crocodiles"

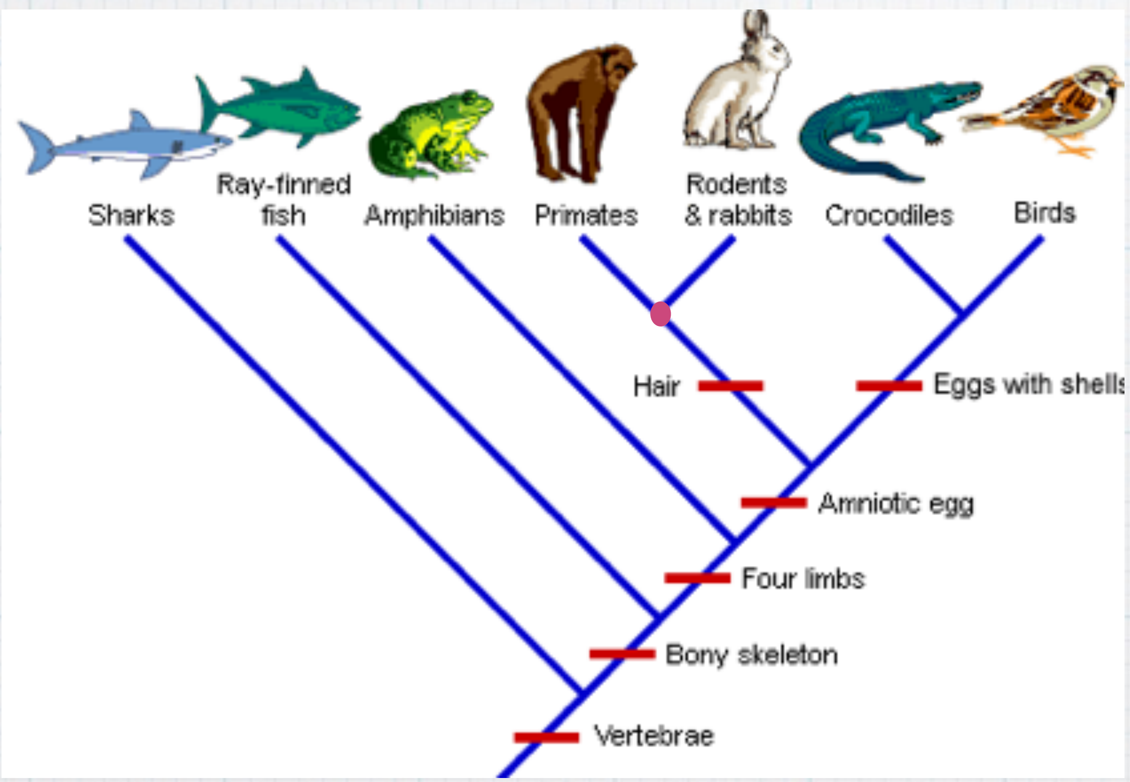
Is **F** true in this interpretation?

Semantics of ∀ and ∃

Extending I(F) to formulas with quantifiers:

* **Forall**: I(∀x:F) = 1 iff I[a/x](F) = 1 for all a in D
* **Exists**: I(∃x: F) = 1 iff I[a/x](F) = 1 for some a in D

Example 1

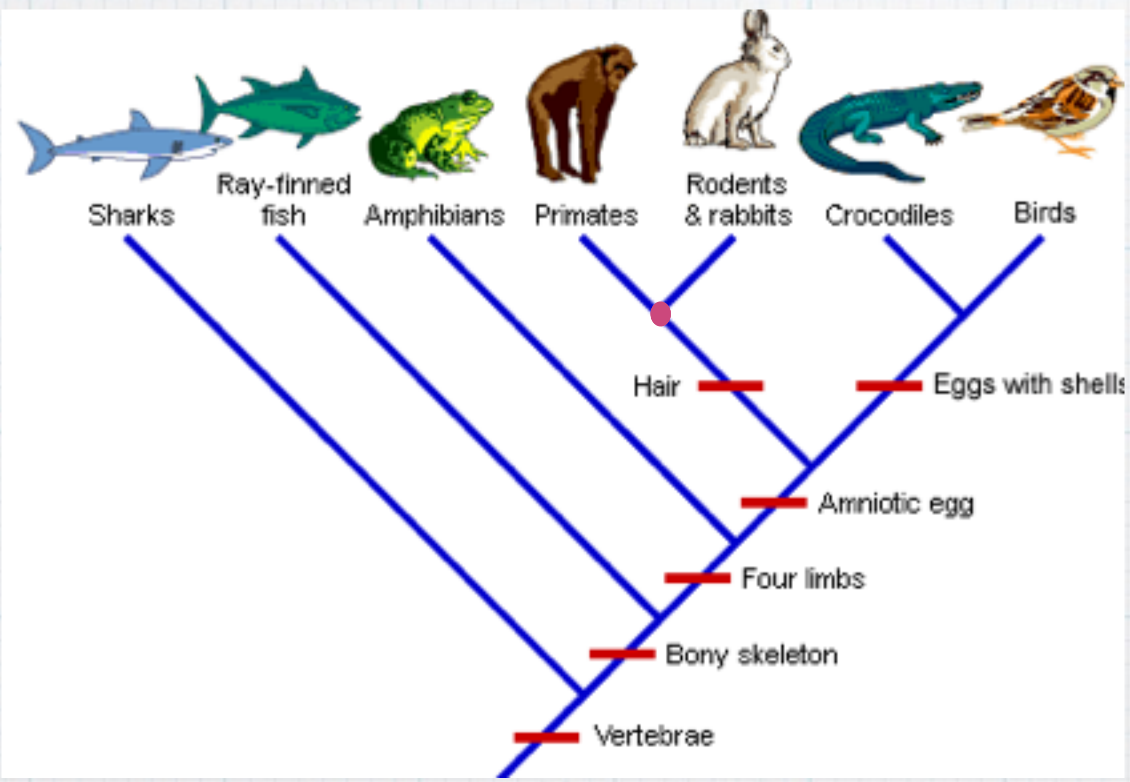


F: $\exists x,y,z(z < x \wedge z < y)$

Interpretation:
left phylogeny tree

Is **F** true in this interpretation?

Example 2



F: $\forall x,y,z(x<y \wedge y<z \rightarrow x<z)$

Interpretation:
left phylogeny tree

Is **F** true in this interpretation?

Exercise 1

Formally express that every two species have a common ancestor. Show that this is true in the phylogeny tree interpretation.

Exercise 2

Consider the following interpretation (social network):

Relations: **Friends/2**

D = {people}

I: Friends = {(x,y) : x is a friend of y }

Express (the famous) six-degree of separation:
"Every two people have distance six in this graph"

Exercise 3

In the linear arithmetic **(N,+)** model, argue the following formulas are true:

- $\forall x \exists y : y > x$
- $\forall x \exists y : y + y = x \vee y + y + 1 = x$

Exercise 4

Consider the interpretation:

D = {0,1,...,8}

I: R -> { (x,y) : y = x - z, z=1,2,3 }

Prove that the formula is true:

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 (R(x_1, y_1) \wedge R(y_1, x_2) \wedge R(x_2, y_2) \wedge R(y_2, 0))$$

Exercise 5

Consider the interpretation:

D = {integer}

I: R -> { (x,y) : y = x - z, z=1,2,3 }

Prove that the formula below is true:

$$\forall x((\exists w : 4w = x) \rightarrow \forall z \exists y(R(x,z) \rightarrow R(z,y) \wedge (\exists w : 4w = y)))$$

Note: **4w** is a "macro" for **w+w+w+w**
(even this is a macro)

Satisfiability/ Validity/Equivalence

Satisfiability/Validity/(Semantic) Equivalence

* A formula is **satisfiable** if it is true in some interpretation
* A formula is **valid** if it is true in all interpretations
* Two formulas are **equivalent** if their truth values are the same under all interpretations

Exercises

Show that all the following examples are satisfiable!

$$\exists x : \text{man}(x) \wedge \neg\text{married}(x)$$

$$(\exists x : \text{man}(x)) \rightarrow (\forall y : \text{man}(x))$$

$$(\forall x : \text{man}(x)) \rightarrow (\exists y : \text{man}(x))$$

$$\forall x : \text{man}(x) \rightarrow \text{man}(x)$$

Exercises

Point out valid and invalid formulas!

$$\exists x : \text{man}(x) \wedge \neg\text{married}(x)$$

$$(\exists x : \text{man}(x)) \rightarrow (\forall y : \text{man}(x))$$

$$(\forall x : \text{man}(x)) \rightarrow (\exists y : \text{man}(x))$$

$$\forall x : \text{man}(x) \rightarrow \text{man}(x)$$

More exercises

Prove that the following formulas are valid

$$(\forall x(\mathrm{Man}(x) \rightarrow \mathrm{Mortal}(x)) \wedge \mathrm{Man}(Socrates)) \rightarrow \mathrm{Mortal}(Socrates)$$

$$\forall x(P(x)) \rightarrow \forall y(P(y))$$

Prove that the following formula is not valid

$$((\exists x : P(x)) \wedge (\exists x : R(x)) \rightarrow (\exists x : P(x) \wedge R(x))$$

Some equivalences

* Equivalences from boolean logic carry over to FOL
* New ones, eg, De Morgan's Laws for FOL:

$$\neg \exists x \neg F \equiv \forall x F$$

$$\neg \forall x \neg F \equiv \exists x F$$

Exercise

Prove De Morgan's Laws!

Ponderables

* What's the connection between satisfiability/validity/equivalence?
* Could you give an algorithm for checking satisfiability/validity/equivalence?
* What about the same problem over "finite interpretations"? Over "finite interpretations of size k"?

Roadmap for FOL after this

* Quantifier-free FOL
* Specialised interpretations: linear arithmetic, theory of strings (?), theory of arrays (?), ...

Some more tutorial questions

Free variables

Define this by induction on formula F:

- $free(R(x,y,c)) = \{x,y\}$
- $free(F \wedge F') = free(F) \cup free(F')$
- $free(\neg F) = free(F)$
- $free(\forall x{:}F) = free(F) - \{x\}$
- $free(\exists x{:}F) = free(F) - \{x\}$

Exercises

What are the free variables of the formulas:

$$\exists x : even(x)$$

$$(\forall x : R(x)) \wedge Z(x)$$

More equivalences

If **x** is not free in the formula **G**, then:

$$(\forall x : F) \wedge G \equiv \forall x(F \wedge G)$$

$$(\exists x : F) \wedge G \equiv \exists x(F \wedge G)$$