Logic: Sample solutions to Homework 2

柯向上

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1. Let $\Gamma : \textsc{List Prop}$ and $\varphi : \textsc{Prop}$. Prove that $\Gamma \models \varphi$ if and only if the list $\Gamma, \neg\varphi$ is not satisfiable.

ASSUME $\Gamma : \textsc{List Prop}$, $\varphi : \textsc{Prop}$

PROVE $\Gamma \models \varphi$ if and only if $\Gamma, \neg\varphi$ is not satisfiable

PROOF

1 $\Gamma \models \varphi$ only if $\Gamma, \neg\varphi$ is not satisfiable.

ASSUME $\Gamma \models \varphi$, $\sigma : \mathcal{PV} \to \mathbf{2}$, σ satisfies $\Gamma, \neg\varphi$

PROVE Contradiction

PROOF

1.1 $[\![\varphi]\!]_\sigma = 1$.

PROOF The assumption that σ satisfies $\Gamma, \neg\varphi$ implies that σ satisfies Γ, which, together with the assumption $\Gamma \models \varphi$, implies that σ satisfies φ.

1.2 $[\![\neg\varphi]\!]_\sigma = 1$.

PROOF The assumption that σ satisfies $\Gamma, \neg\varphi$ implies that σ satisfies $\neg\varphi$.

1.3 $[\![\varphi]\!]_\sigma = 0$.

PROOF $\boxed{1.2}$ is equivalent to $[\![\varphi]\!]_\sigma = 0$ by the truth-value interpretation of negation.

1.4 Contradiction (QED).

PROOF $\boxed{1.1}$ and $\boxed{1.3}$ implies $0 = 1$, an apparent contradiction.

2 $\Gamma \models \varphi$ if $\Gamma, \neg\varphi$ is not satisfiable.

ASSUME $\Gamma, \neg\varphi$ is not satisfiable

PROVE $\Gamma \models \varphi$

ASSUME $\sigma : \mathcal{PV} \to \mathbf{2}$, σ satisfies Γ

PROVE $[\![\varphi]\!]_\sigma = 1$

PROOF

2.1 $[\![\neg\varphi]\!]_\sigma \neq 1$.

ASSUME $[\![\neg\varphi]\!]_\sigma = 1$

PROVE Contradiction

2.1.1 σ satisfies $\Gamma, \neg\varphi$.

PROOF σ satisfies both Γ and $\neg\varphi$ by assumption.

2.1.2 Contradiction (QED).

PROOF $\boxed{2.1.1}$ means that the list $\Gamma, \neg\varphi$ is satisfiable, contradicting the assumption that it is not satisfiable.

2.2 $[\![\neg\varphi]\!]_\sigma = 0$.

PROOF Either $[\![\neg\varphi]\!]_\sigma = 0$ or $[\![\neg\varphi]\!]_\sigma = 1$; the latter case is shown to be impossible by $\boxed{2.1}$.

2.3 $[\![\varphi]\!]_\sigma = 1$ (QED).

PROOF 2.2 and the truth-value interpretation of negation.

3 QED.

PROOF The two directions are proved in 1 and 2.

2. How do you check satisfiability of a propositional formula and "finite" semantic consequence (i.e., a statement of the form $\Gamma \models \varphi$ where the number of propositional formulas in Γ is finite) using the truth table method? Justify your answer.

   Solution. To show satisfiability of a propositional formula φ: let x_1, ..., x_n be the propositional variables occurring in φ. Construct a truth table by listing the 2^n possible truth-value assignments to the n variables and computing the truth-value of φ under these assignments. If any of the assignments makes the truth-value of φ compute to 1, then φ is satisfiable, since, for instance, the valuation $\sigma_0[v_1/x_1] \ldots [v_n/x_n]$ satisfies φ, where σ_0 is the valuation that is everywhere 0, and v_1, ..., v_n are the truth-values assigned to the variables x_1, ..., x_n by that assignment. Otherwise, if none of the assignments makes the truth-value of φ compute to 1, then φ is unsatisfiable, since, for any valuation σ, $[\![\varphi]\!]_\sigma$ must be equal to one of the results computed in the truth table and is hence 0.

   For "finite" semantic consequence, say $\varphi_1, \ldots, \varphi_n \models \psi$, observe that it is equivalent to $\models \varphi_1 \to \cdots \to \varphi_n \to \psi$, so it suffices to check the validity of the propositional formula $\varphi_1 \to \cdots \to \varphi_n \to \psi$ by a truth table. The equivalence ($\varphi_1, \ldots, \varphi_n \models \psi$ if and only if $\models \varphi_1 \to \cdots \to \varphi_n \to \psi$) can be proved by induction on n.

   (Note: it is possible to make the above justifications more rigorous, but we do not take the trouble to do so here.)

3. Show that $\models \neg\neg\mathtt{A} \to \mathtt{A}$.

   Solution. See the following truth table:

A	¬	¬	A	→	A
0	0	1	0	1	0
1	1	0	1	1	1

4. Show that $\models ((\mathtt{A} \to \mathtt{B}) \to \mathtt{A}) \to \mathtt{A}$.

   Solution. See the following truth table:

A	B	((A	→	B)	→	A)	→	A
0	0	0	1	0	0	0	1	0
0	1	0	1	1	0	0	1	0
1	0	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1	1

5. Prove statement 1.4 in the proof of the soundness theorem.

1.4 ASSUME $\Gamma : \text{List Prop}^-,\ \varphi, \psi : \text{Prop}^-,$
$\Gamma \vdash_{\mathrm{NJ}^-} \varphi \to \psi,\ \Gamma \models \varphi \to \psi,\ \Gamma \vdash_{\mathrm{NJ}^-} \varphi,\ \Gamma \models \varphi$

PROVE $\Gamma \models \psi$

ASSUME $\sigma : \mathcal{PV} \to \mathbf{2},\ \sigma$ satisfies Γ

PROVE $[\![\psi]\!]_\sigma = 1$

PROOF

1.4.1 $[\![\varphi \to \psi]\!]_\sigma = 1$.

PROOF $\Gamma \models \varphi \to \psi$ and σ satisfies Γ.

1.4.2 $[\![\varphi]\!]_\sigma \leq [\![\psi]\!]_\sigma$.

PROOF 1.4.1 and the truth-value interpretation of '→'.

1.4.3 $[\![\varphi]\!]_\sigma = 1$.

PROOF $\Gamma \models \varphi$ and σ satisfies Γ.

1.4.4 $[\![\psi]\!]_\sigma = 1$ (QED).

PROOF Either $[\![\psi]\!]_\sigma = 0$ or $[\![\psi]\!]_\sigma = 1$. $\boxed{1.4.2}$ and $\boxed{1.4.3}$ together imply that $1 \leq [\![\psi]\!]_\sigma$, so $[\![\psi]\!]_\sigma$ is necessarily 1.

6. Prove the weakening lemma.

   Contact me if you wish to see the proof.

7. Prove statement 3.2 in the proof of the reconstruction lemma.

3 ASSUME $\varphi : \textsc{Prop}^-,\ T_\sigma\ (\mathit{vars}\ \varphi) \vdash_{\mathrm{NK}^-} T_\sigma\ \varphi$,
$\psi : \textsc{Prop}^-,\ T_\sigma\ (\mathit{vars}\ \psi) \vdash_{\mathrm{NK}^-} T_\sigma\ \psi$

PROVE $T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)) \vdash_{\mathrm{NK}^-} T_\sigma\ (\varphi \to \psi)$

PROOF Case analysis to determine $T_\sigma\ (\varphi \to \psi)$.

3.2 CASE $[\![\psi]\!]_\sigma = 0$

PROOF

3.2.1 CASE $[\![\varphi]\!]_\sigma = 0$

PROOF In this case we need to produce a derivation of $\varphi \to \psi$.

3.2.1.1 $T_\sigma\ (\mathit{vars}\ \varphi) \vdash_{\mathrm{NK}^-} \neg\varphi$

PROOF Induction hypothesis $T_\sigma\ (\mathit{vars}\ \varphi) \vdash_{\mathrm{NK}^-} T_\sigma\ \varphi$, where $T_\sigma\ \varphi = \neg\varphi$ because of the assumption $[\![\varphi]\!]_\sigma = 0$.

3.2.1.2 LET $d : \mathrm{NK}^-[T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)),\ \varphi; \neg\varphi]$

PROOF $\boxed{3.2.1.1}$ and weakening.

3.2.1.3 $T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)) \vdash_{\mathrm{NK}^-} \varphi \to \psi$ (QED).

PROOF

$$\dfrac{\dfrac{\dfrac{d \qquad \dfrac{}{T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)),\ \varphi \vdash \varphi}\,(\text{assum})}{T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)),\ \varphi \vdash \bot}\,(\to\text{E})}{T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)),\ \varphi \vdash \psi}\,(\bot\text{E})}{T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)) \vdash \varphi \to \psi}\,(\to\text{I})$$

3.2.2 CASE $[\![\varphi]\!]_\sigma = 1$

PROOF In this case we need to produce a derivation of $\neg(\varphi \to \psi)$.

3.2.2.1 $T_\sigma\ (\mathit{vars}\ \varphi) \vdash_{\mathrm{NK}^-} \varphi$

PROOF Induction hypothesis $T_\sigma\ (\mathit{vars}\ \varphi) \vdash_{\mathrm{NK}^-} T_\sigma\ \varphi$, where $T_\sigma\ \varphi = \varphi$ because of the assumption $[\![\varphi]\!]_\sigma = 1$.

3.2.2.2 LET $d : \mathrm{NK}^-[T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)),\ \varphi \to \psi; \varphi]$

PROOF $\boxed{3.2.2.1}$ and weakening.

3.2.2.3 $T_\sigma\ (\mathit{vars}\ \psi) \vdash_{\mathrm{NK}^-} \neg\psi$

PROOF Induction hypothesis $T_\sigma\ (\mathit{vars}\ \psi) \vdash_{\mathrm{NK}^-} T_\sigma\ \psi$, where $T_\sigma\ \psi = \neg\psi$ because of the assumption $[\![\psi]\!]_\sigma = 0$.

3.2.2.4 LET $e : \mathrm{NK}^-[T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)),\ \varphi \to \psi; \neg\psi]$

PROOF $\boxed{3.2.2.3}$ and weakening.

3.2.2.5 QED.

PROOF

$$\dfrac{\dfrac{e \qquad \dfrac{\dfrac{}{T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)),\ \varphi \to \psi \vdash \varphi \to \psi}\,(\text{assum}) \qquad d}{T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)),\ \varphi \to \psi \vdash \psi}\,(\to\text{E})}{T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)),\ \varphi \to \psi \vdash \bot}\,(\to\text{E})}{T_\sigma\ (\mathit{vars}\ (\varphi \to \psi)) \vdash \neg(\varphi \to \psi)}\,(\to\text{I})$$

3.2.3 QED.

PROOF Either $[\![\varphi]\!]_\sigma = 0$ or $[\![\varphi]\!]_\sigma = 1$; $\boxed{3.2.1}$ and $\boxed{3.2.2}$.

8. Give the derivation of $\vdash_{\mathrm{NK}^-} \mathtt{A} \to \mathtt{A}$ constructed by the completeness theorem.

   Contact me if you wish to see the derivation.

9. Let $\varphi := (\forall\,\mathtt{x.\ P\ x}) \to (\exists\,\mathtt{x.\ P\ x})$. Show that $\vdash_{\mathrm{NJ}} \varphi$ and $\models \varphi$.

   Solution. For derivability of φ:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall\,\mathtt{x.\ P\ x} \vdash \forall\,\mathtt{x.\ P\ x}}\,(\text{assum})}{\forall\,\mathtt{x.\ P\ x} \vdash \mathtt{P\ x}}\,(\forall\text{E})}{\forall\,\mathtt{x.\ P\ x} \vdash \exists\,\mathtt{x.\ P\ x}}\,(\exists\text{I})}{\vdash (\forall\,\mathtt{x.\ P\ x}) \to (\exists\,\mathtt{x.\ P\ x})}\,(\to\text{I})$$

   For validity of φ: for any structure $\mathcal{M}$ and $\mathcal{M}$-assignment σ,

$$\begin{array}{ll} & [\![(\forall\,\mathtt{x.\ P\ x}) \to (\exists\,\mathtt{x.\ P\ x})]\!]_{\mathcal{M},\,\sigma} = 1 \\ \Leftrightarrow & \quad \{\text{truth value of `}\to\text{'}\} \\ & [\![\forall\,\mathtt{x.\ P\ x}]\!]_{\mathcal{M},\,\sigma} \leq [\![\exists\,\mathtt{x.\ P\ x}]\!]_{\mathcal{M},\,\sigma}. \qquad (*) \end{array}$$

   Either $[\![\forall\,\mathtt{x.\ P\ x}]\!]_{\mathcal{M},\,\sigma} = 0$ or $[\![\forall\,\mathtt{x.\ P\ x}]\!]_{\mathcal{M},\,\sigma} = 1$. In the first case, condition $(*)$ is necessarily true. In the second case, we reason:

$$\begin{array}{ll} & [\![\forall\,\mathtt{x.\ P\ x}]\!]_{\mathcal{M},\,\sigma} = 1 \\ \Leftrightarrow & \quad \{\text{truth value of `}\forall\text{'}\} \\ & [\![\mathtt{P\ x}]\!]_{\mathcal{M},\,\sigma[m/\mathtt{x}]} = 1 \;\; \text{for every } m : \mathcal{M} \\ \Rightarrow & \quad \{\mathcal{M} \text{ is nonempty}\} \\ & [\![\mathtt{P\ x}]\!]_{\mathcal{M},\,\sigma[m/\mathtt{x}]} = 1 \;\; \text{for some } m : \mathcal{M} \\ \Leftrightarrow & \quad \{\text{truth value of `}\exists\text{'}\} \\ & [\![\exists\,\mathtt{x.\ P\ x}]\!]_{\mathcal{M},\,\sigma[m/\mathtt{x}]} = 1, \end{array}$$

   so condition $(*)$ is again necessarily true.

   Discussion. Validity of φ depends essentially on the assumption that the domain is nonempty. This is reflected in the $(\forall\text{E})$ step in the derivation of φ, where we are allowed to instantiate the universally quantified formula at the free variable $\mathtt{x}$.

10. Prove *Glivenko's Theorem*: $\Gamma \vdash_{\mathrm{NK}} \varphi$ if and only if $\neg\neg\Gamma \vdash_{\mathrm{NJ}} \neg\neg\varphi$ for every $\Gamma : \textsc{List Prop}$ and $\varphi : \textsc{Prop}$, where $\neg\neg\Gamma := [\,\neg\neg\varphi \mid \varphi \in \Gamma\,]$.

    Contact me if you wish to see the proof. The point of this theorem is that, to embed classical propositional logic into intuitionistic propositional logic, it suffices to put double negation in front of a classical propositional formula. This can be seen as a simplification of the Gödel–Gentzen negative translation for propositional logic. The simplification is not valid for first-order logic due to the presence of quantifiers.