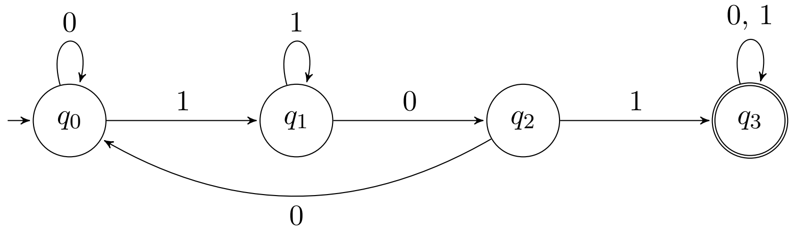
Suggested Solutions to Homework Assignment #2

[Compiled on July 7, 2011]

1. (20 Points) For each of the following regular languages, draw the state diagram of a DFA that recognizes the language.

   (a) $\{w \in \{0,1\}^* \mid w \text{ contains } 101 \text{ as a substring}\}$.

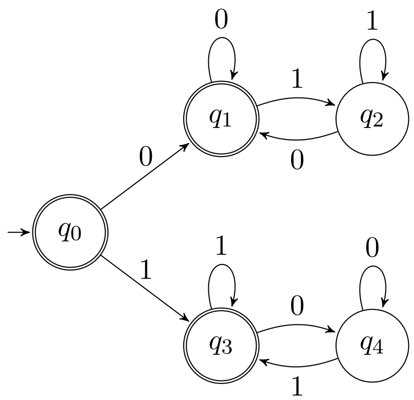
   Solution.



□

   (b) $\{w \in \{0,1\}^* \mid w \text{ has equal occurrences of } 01 \text{ and } 10 \text{ as substrings}\}$.

   Solution.

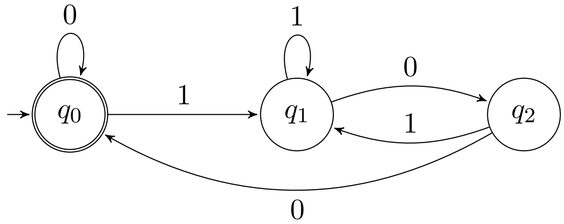


□

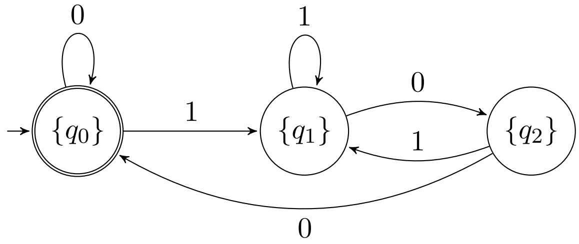
2. (20 Points) Draw the state diagram of an NFA *with three states* that recognizes the language $\{w \in \{0,1\}^* \mid w \text{ is a multiple of } 4\}$ and then convert it into a DFA using the subset construction (showing the states of the DFA as sets of states of the NFA).

   Solution.

   NFA:

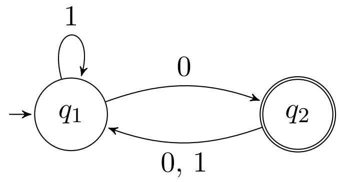


DFA:



$\square$

3. (20 Points) Find a regular expression as short as possible that describes the language of the following DFA.



Solution. 1*0((0 ∪ 1)1*0)* $\square$

4. (20 Points) Below are two facts (the second of which was discussed in class) about context-free languages:

   - The intersection of a context-free language and a regular language is context-free.
   - The language $\{a^n b^n c^n \mid n \geq 0\}$ is not context-free.

   Prove using these facts that the language over $\{a, b, c\}$ with equal numbers of a's, b's, and c's is not context-free.

   Solution. Let A denote the language over $\{a, b, c\}$ with equal numbers of a's, b's, and c's. Let $B = \{a^i b^j c^k \mid i, j, k \geq 0\}$ (or $a^*b^*c^*$), which is regular. As the intersection of a context-free language and a regular language is context-free (first fact), if A were context-free, then $A \cap B$ would also be context-free. However, $A \cap B$ equals $\{a^n b^n c^n \mid n \geq 0\}$, which is not context-free (second fact), and it follows that A is not context-free. $\square$

5. (20 Points) Prove, using the pumping lemma, that $\{a^n \mid n \text{ is a prime number}\}$ is not context-free. (Hint: Suppose p is the pumping length. Consider a string $s = a^{p'}$, where p' is a prime number greater than or equal to p. Consider an arbitrary division of s as $uvxyz = a^i a^j a^k a^l a^{p'-i-j-k-l}$, where $j+l > 0$ ($|vy| > 0$) and $j+k+l \leq p \leq p'$ ($|vxy| \leq p$).)

   Solution. Suppose p is the pumping length. Consider a string $s = a^{p'}$, where p' is a prime number greater than or equal to p. We further suppose that s can be pumped by dividing s as $uvxyz = a^i a^j a^k a^l a^{p'-i-j-k-l}$, where $j+l > 0$ ($|vy| > 0$) and $j+k+l \leq p \leq p'$ ($|vxy| \leq p$). We can pump s up to $a^i (a^j)^m a^k (a^l)^m a^{p'-i-j-k-l}$ for any $m > 1$, obtaining strings of the form $a^{jm+lm+p'-j-l} = a^{(j+l)(m-1)+p'}$. However, for $m = p' + 1$, $a^{(j+l)(m-1)+p'} = a^{(j+l)(p'+1-1)+p'} = a^{(j+l+1)p'}$ is clearly not in the language $\{a^n \mid n \text{ is a prime number}\}$. Thus, s cannot be pumped and the language is not context-free. $\square$