Suggested Solutions to Homework Assignment #1

[Compiled on July 7, 2011]

1. (20 Points) Prove using truth tables the following semantic entailments, where P, Q, and R are proposition symbols.

   (a) $(P \to Q) \land (P \to R) \models P \to (Q \land R)$

   Solution.

P	Q	R	$P \to Q$	$P \to R$	$Q \land R$	$(P \to Q) \land (P \to R)$	$P \to (Q \land R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

   Since every assignment that satisfies $(P \to Q) \land (P \to R)$ also satisfies $P \to (Q \land R)$, $(P \to Q) \land (P \to R) \models P \to (Q \land R)$. □

   (b) $P \lor Q \models \neg(\neg P \land \neg Q)$

   Solution.

P	Q	$\neg P$	$\neg Q$	$\neg P \land \neg Q$	$P \lor Q$	$\neg(\neg P \land \neg Q)$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

   Since every assignment that satisfies $P \lor Q$ also satisfies $\neg(\neg P \land \neg Q)$, $P \lor Q \models \neg(\neg P \land \neg Q)$. □

2. (20 Points) Prove using natural deduction (in the sequent form) the following sequents, where P, Q, and R are proposition symbols.

   (a) $(P \to Q) \land (P \to R) \vdash P \to (Q \land R)$

   Solution.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \to Q) \land (P \to R), P \vdash (P \to Q) \land (P \to R)}\,(Ax)}{(P \to Q) \land (P \to R), P \vdash P \to Q}\,(\land E_1) \quad \dfrac{}{(P \to Q) \land (P \to R), P \vdash P}\,(Ax)}{(P \to Q) \land (P \to R), P \vdash Q}\,(\to E) \quad \alpha}{(P \to Q) \land (P \to R), P \vdash Q \land R}\,(\land I)}{(P \to Q) \land (P \to R) \vdash P \to (Q \land R)}\,(\to I)
$$

α:

$$
\dfrac{\dfrac{\dfrac{}{(P \to Q) \land (P \to R), P \vdash (P \to Q) \land (P \to R)}\,(Ax)}{(P \to Q) \land (P \to R), P \vdash P \to R}\,(\land E_2) \quad \dfrac{}{(P \to Q) \land (P \to R), P \vdash P}\,(Ax)}{(P \to Q) \land (P \to R), P \vdash R}\,(\to E)
$$

$\square$

(b) $P \lor Q \vdash \neg(\neg P \land \neg Q)$

Solution.

$$\dfrac{\dfrac{}{P \lor Q \vdash P \lor Q}\,(Ax) \qquad \alpha \qquad \beta}{P \lor Q \vdash \neg(\neg P \land \neg Q)}\,(\lor E)$$

α:

$$\dfrac{\dfrac{\dfrac{}{P \lor Q, P, \neg P \land \neg Q \vdash P}\,(Ax) \qquad \dfrac{\dfrac{}{P \lor Q, P, \neg P \land \neg Q \vdash \neg P \land \neg Q}\,(Ax)}{P \lor Q, P, \neg P \land \neg Q \vdash \neg P}\,(\land E_1)}{P \lor Q, P, \neg P \land \neg Q \vdash P \land \neg P}\,(\land I)}{P \lor Q, P \vdash \neg(\neg P \land \neg Q)}\,(\neg I)$$

β:

$$\dfrac{\dfrac{\dfrac{}{P \lor Q, Q, \neg P \land \neg Q \vdash Q}\,(Ax) \qquad \dfrac{\dfrac{}{P \lor Q, Q, \neg P \land \neg Q \vdash \neg P \land \neg Q}\,(Ax)}{P \lor Q, Q, \neg P \land \neg Q \vdash \neg Q}\,(\land E_2)}{P \lor Q, Q, \neg P \land \neg Q \vdash Q \land \neg Q}\,(\land I)}{P \lor Q, Q \vdash \neg(\neg P \land \neg Q)}\,(\neg I)$$

$\square$

3. (20 Points) Consider the structure $\mathcal{N} = (\mathrm{N}, \{+, \times, 0, 1, <\})$, i.e., the set of natural numbers with the usual functions, constants, and predicates ("$=$" is implicitly assumed).

   (a) Write a first-order formula to define the set of odd numbers (i.e., a formula with a free variable such that the formula is true exactly when the free variable is assigned an odd number).

   Solution.

   $$\exists y(x = y + y + 1)$$

   $\square$

   (b) Write a first-order formula to define the set of prime numbers.

   Solution.

   $$1 < x \land \forall y \forall z(x = y \times z \to (y = 1 \lor z = 1))$$

   $\square$

4. (10 Points) Consider the set of natural numbers with addition $(\mathrm{N}, \{+\})$ and the set of integers with addition $(\mathrm{Z}, \{+\})$. Give a first-order sentence that is true in one but false in the other. (Note: two structures are said to be *elementarily equivalent* if they satisfy the same set of first-order sentences. So, the sentence you would give shows that $(\mathrm{N}, \{+\})$ and $(\mathrm{Z}, \{+\})$ are not elementarily equivalent.)

   Solution.
   Since all natural numbers are non-negative, it is impossible that for any two natural numbers x and y, there exists a natural number z satisfying $x + z = y$. Thus the following formula is true in $(\mathrm{Z}, \{+\})$ but false in $(\mathrm{N}, \{+\})$.

   $$\forall x \forall y \exists z(x + z = y)$$

   $\square$

5. (30 Points) Prove using natural deduction (in the sequent form) the following sequents, where p, q, and r are unary predicates. (Note: proofs of 5a and 5b would complete the proof on Slide 56 of the lecture note for Elementary Logic.)

   (a) $\forall x(\neg p(x) \lor \neg q(x)) \land \forall x(r(x) \to q(x)), p(y), r(y) \vdash r(y) \to q(y)$

   Solution.

$$\dfrac{\dfrac{\dfrac{}{\forall x(\neg p(x) \lor \neg q(x)) \land \forall x(r(x) \to q(x)), p(y), r(y) \vdash \forall x(\neg p(x) \lor \neg q(x)) \land \forall x(r(x) \to q(x))}\,(Ax)}{\forall x(\neg p(x) \lor \neg q(x)) \land \forall x(r(x) \to q(x)), p(y), r(y) \vdash \forall x(r(x) \to q(x))}\,(\land E)}{\forall x(\neg p(x) \lor \neg q(x)) \land \forall x(r(x) \to q(x)), p(y), r(y) \vdash r(y) \to q(y)}\,(\forall E)$$

$\square$

   (b) $\forall x(\neg p(x) \lor \neg q(x)) \land \forall x(r(x) \to q(x)), p(y), r(y) \vdash \neg q(y)$

   Solution. Let γ denote $\forall x(\neg p(x) \lor \neg q(x)) \land \forall x(r(x) \to q(x))$. We will omit $r(y)$ in the antecedent (to save space), as it will not be used.

$$\dfrac{\alpha \qquad \dfrac{\dfrac{}{\gamma, p(y), \neg p(y) \vdash p(y)}\,(Ax) \qquad \dfrac{}{\gamma, p(y), \neg p(y) \vdash \neg p(y)}\,(Ax)}{\gamma, p(y), \neg p(y) \vdash \neg q(y)}\,(\neg E) \qquad \dfrac{}{\gamma, p(y), \neg q(y) \vdash \neg q(y)}\,(Ax)}{\gamma, p(y) \vdash \neg q(y)}\,(\lor E)$$

   α:

$$\dfrac{\dfrac{\dfrac{}{\forall x(\neg p(x) \lor \neg q(x)) \land \forall x(r(x) \to q(x)), p(y) \vdash \forall x(\neg p(x) \lor \neg q(x)) \land \forall x(r(x) \to q(x))}\,(Ax)}{\forall x(\neg p(x) \lor \neg q(x)) \land \forall x(r(x) \to q(x)), p(y) \vdash \forall x(\neg p(x) \lor \neg q(x))}\,(\land E)}{\forall x(\neg p(x) \lor \neg q(x)) \land \forall x(r(x) \to q(x)), p(y) \vdash \neg p(y) \lor \neg q(y)}\,(\forall E)$$

$\square$

   (c) $p(y) \vdash \forall x(x = y \to p(x))$

   Solution.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(y), z = y \vdash z = y}\,(Ax) \qquad \dfrac{}{p(y), z = y \vdash z = z \ ((x = z)[z/x])}\,(= I)}{p(y), z = y \vdash y = z \ ((x = z)[y/x])}\,(= E) \qquad \dfrac{}{p(y), z = y \vdash p(y)}\,(Ax)}{p(y), z = y \vdash p(z)}\,(= E)}{p(y) \vdash z = y \to p(z)}\,(\to I)}{p(y) \vdash \forall x(x = y \to p(x))}\,(\forall I)$$

$\square$