Logic
Part III: Medley

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Recap: Sequent Calculus

Example 1

$$\neg\forall x.P(x) \vdash \exists x.\neg P(x)$$

Example 1

$$(\neg\mathrm{L})\ \frac{\vdash \forall x.P(x), \exists x.\neg P(x)}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}$$

Example 1

$$
\dfrac{\dfrac{\vdash P(x), \exists x.\neg P(x)}{\vdash \forall x.P(x), \exists x.\neg P(x)}\;(\forall\mathrm{R})}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\;(\neg\mathrm{L})
$$

Example 1

$$
\dfrac{\dfrac{\dfrac{\vdash P(x), \neg P(x)[x/x]}{\vdash P(x), \exists x.\neg P(x)}\,(\exists\mathrm{R})}{\vdash \forall x.P(x), \exists x.\neg P(x)}\,(\forall\mathrm{R})}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\,(\neg\mathrm{L})
$$

Example 1

$$
\dfrac{\dfrac{\dfrac{\vdash P(x), \neg P(x)}{\vdash P(x), \exists x.\neg P(x)}\,(\exists\mathrm{R})}{\vdash \forall x.P(x), \exists x.\neg P(x)}\,(\forall\mathrm{R})}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\,(\neg\mathrm{L})
$$

Example 1

$$
\dfrac{(\exists\mathrm{R})\;\dfrac{(\neg\mathrm{R})\;\dfrac{P(x) \vdash P(x)}{\vdash P(x), \neg P(x)}}{\vdash P(x), \exists x.\neg P(x)}}{}
$$

$$
\begin{array}{rc}
(\neg\mathrm{R}) & \dfrac{P(x) \vdash P(x)}{\vdash P(x), \neg P(x)} \\
(\exists\mathrm{R}) & \dfrac{}{\vdash P(x), \exists x.\neg P(x)} \\
(\forall\mathrm{R}) & \dfrac{}{\vdash \forall x.P(x), \exists x.\neg P(x)} \\
(\neg\mathrm{L}) & \dfrac{}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}
\end{array}
$$

Example 2: Drinker Paradox with Cut

$$
\text{(cut)}\ \dfrac{
\text{(}\lor\text{R)}\ \dfrac{
\text{(}\neg\text{R)}\ \dfrac{\forall x.D(x) \vdash \forall x.D(x), \exists x.D(x) \to \forall y.D(y)}{\vdash \forall x.D(x), \neg\forall x.D(x), \exists x.D(x) \to \forall y.D(y)}
}{\vdash (\forall x.D(x)) \lor (\neg\forall x.D(x)), \exists x.D(x) \to \forall y.D(y)}
\qquad
\text{(}\lor\text{L)}\ \dfrac{
\text{(}\exists\text{R)}\ \dfrac{
\text{(}\to\text{R)}\ \dfrac{
\text{(}\forall\text{R)}\ \dfrac{
\text{(}\forall\text{L)}\ \dfrac{D(y), D(x) \vdash D(y)}{\forall x.D(x), D(x) \vdash D(y)}
}{\forall x.D(x), D(x) \vdash \forall y.D(y)}
}{\forall x.D(x) \vdash D(x) \to \forall y.D(y)}
}{\forall x.D(x) \vdash \exists x.D(x) \to \forall y.D(y)}
\quad
\text{(}\neg\text{L)}\ \dfrac{
\text{(}\forall\text{R)}\ \dfrac{
\text{(}\exists\text{R)}\ \dfrac{
\text{(}\to\text{R)}\ \dfrac{D(x) \vdash D(x), \forall y.D(y)}{\vdash D(x), D(x) \to \forall y.D(y)}
}{\vdash D(x), \exists x.D(x) \to \forall y.D(y)}
}{\vdash \forall x.D(x), \exists x.D(x) \to \forall y.D(y)}
}{\neg\forall x.D(x) \vdash \exists x.D(x) \to \forall y.D(y)}
}{(\forall x.D(x)) \lor (\neg\forall x.D(x)) \vdash \exists x.D(x) \to \forall y.D(y)}
}{\vdash \exists x.D(x) \to \forall y.D(y)}
$$

Example 3: Drinker Paradox without Cut

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D(x), D(y) \vdash D(y), \forall y.D(y)}{D(x) \vdash D(y), D(y) \to \forall y.D(y)}\ (\to\mathrm{R})}{D(x) \vdash D(y), \exists x.D(x) \to \forall y.D(y)}\ (\exists\mathrm{R})}{D(x) \vdash \forall y.D(y), \exists x.D(x) \to \forall y.D(y)}\ (\forall\mathrm{R})}{\vdash D(x) \to \forall y.D(y), \exists x.D(x) \to \forall y.D(y)}\ (\to\mathrm{R})}{\vdash \exists x.D(x) \to \forall y.D(y)}\ (\exists\mathrm{R})
$$

Peano Arithmetic

皮亞諾算術

Principles of Arithmetic

- arithmetic $\approx$ calculating with natural numbers
- early logical axiomatisation given in 1889 by Giuseppe Peano (1858–1932)
- we use slightly modernised formulation

Why an Axiomatisation?

- Quiz: Why is $x + y = y + x$?
- three possible answers:
    - Don't you have anything better to worry about?
    - We can model numbers and addition in set theory, then prove that $x + y = y + x$ by using laws of set theory.
    - We can show that it follows from some very simple axioms.

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The Peano Axioms

- signature of arithmetic: $\Sigma_A := \langle\{\mathbf{0}/0, s/1, +/2, \times/2\}, \emptyset\rangle$
- first-order theory of arithmetic: smallest set T_A containing
  1. $\forall x.\forall y.s(x) \doteq s(y) \rightarrow x \doteq y$
  2. $\forall x.\neg(s(x) \doteq \mathbf{0})$
  3. $\forall x.x + \mathbf{0} \doteq x$
  4. $\forall x.\forall y.x + s(y) \doteq s(x + y)$
  5. $\forall x.x \times \mathbf{0} \doteq \mathbf{0}$
  6. $\forall x.\forall y.x \times s(y) \doteq (x \times y) + x$
  7. for any formula φ:
     $\varphi[\mathbf{0}/x] \rightarrow (\forall x.\varphi \rightarrow \varphi[s(x)/x]) \rightarrow (\forall x.\varphi)$

Every natural number n can be represented by a term $\underline{n}$ over Σ_A, as n applications of s to $\mathbf{0}$. For instance, $\underline{2} = s(s(\mathbf{0}))$.

The Standard Model of Arithmetic

$\mathcal{M}_A = \langle \mathbb{N}, \langle [\![\ ]\!]_{\mathtt{F}}, [\![\ ]\!]_{\mathcal{R}} \rangle \rangle$, where

- $[\![\mathbf{0}]\!]_{\mathtt{F}} = 0$
- $[\![s]\!]_{\mathtt{F}}(n) = n + 1$
- $[\![+]\!]_{\mathtt{F}}(m, n) = m + n$
- $[\![\times]\!]_{\mathtt{F}}(m, n) = m \times n$

Standard Model

$\mathcal{M}_A \models T_A$

$\mathcal{M}_A$ is not the only model of T_A.

Relations

- Σ_A does not contain relation symbols
- we can encode $s \leq t$ as $\exists x.s + x \doteq t$, where $x \notin \mathrm{FV}(s) \cup \mathrm{FV}(t)$, because

$$
\begin{array}{ll}
 & \mathcal{M}_A, \sigma \models \exists x.s + x \doteq t \\
\text{iff} & \text{there is } n \in \mathbb{N} \text{ such that } \mathcal{M}_A, \sigma[x := n] \models s + x \doteq t \\
\text{iff} & \text{there is } n \in \mathbb{N} \text{ s. t. } [\![s + x]\!]_{\mathcal{M}_A, \sigma[x:=n]} = [\![t]\!]_{\mathcal{M}_A, \sigma[x:=n]} \\
\text{iff} & \text{there is } n \in \mathbb{N} \text{ s. t. } [\![s]\!]_{\mathcal{M}_A, \sigma[x:=n]} + [\![x]\!]_{\mathcal{M}_A, \sigma[x:=n]} = [\![t]\!]_{\mathcal{M}_A, \sigma[x:=n]} \\
\text{iff} & \text{there is } n \in \mathbb{N} \text{ s. t. } [\![s]\!]_{\mathcal{M}_A, \sigma} + n = [\![t]\!]_{\mathcal{M}_A, \sigma} \\
\text{iff} & [\![s]\!]_{\mathcal{M}_A, \sigma} \leq [\![t]\!]_{\mathcal{M}_A, \sigma}
\end{array}
$$

- $s < t$ is $s \leq t \land \neg(s \doteq t)$, $s \geq t$ is $t \leq s$, and $s > t$ is $t < s$

$T_A \models \forall x.\mathbf{0} + x \doteq x$

$$
\begin{aligned}
T'_A \quad := \quad & \{\forall x.x + \mathbf{0} \doteq x, \forall x.\forall y.x + s(y) \doteq s(x + y), \\
& \mathbf{0} + \mathbf{0} \doteq \mathbf{0} \to (\forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x)) \\
& \quad \to (\forall x.\mathbf{0} + x \doteq x)\}
\end{aligned}
$$

$$
{\scriptstyle(\to\mathrm{L})}\;\dfrac{
{\scriptstyle(\forall\mathrm{L})}\;\dfrac{T'_A, \mathbf{0} + \mathbf{0} \doteq \mathbf{0} \vdash \mathbf{0} + \mathbf{0} \doteq \mathbf{0}, \forall x.\mathbf{0} + x \doteq x}{T'_A \vdash \mathbf{0} + \mathbf{0} \doteq \mathbf{0}, \forall x.\mathbf{0} + x \doteq x}
\qquad
{\scriptstyle(\to\mathrm{L})}\;\dfrac{
{\scriptstyle(\forall\mathrm{R})}\;\dfrac{
{\scriptstyle(\to\mathrm{R})}\;\dfrac{
{\scriptstyle(\mathrm{SUBST})}\;\dfrac{
{\scriptstyle(\forall\mathrm{L})}\;\dfrac{
{\scriptstyle(\forall\mathrm{L})}\;\dfrac{
{\scriptstyle(\mathrm{SUBST})}\;\dfrac{T'_A, \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq \mathbf{0} + s(x), \forall x.\mathbf{0} + x \doteq x}{T'_A, \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}
}{T'_A, \forall y.\mathbf{0} + s(y) \doteq s(\mathbf{0} + y), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}
}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}
}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}
}{T'_A \vdash \mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}
}{T'_A \vdash \forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}
\qquad
T'_A, \forall x.\mathbf{0} + x \doteq x \vdash \forall x.\mathbf{0} + x \doteq x
}{T'_A, (\forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x)) \to (\forall x.\mathbf{0} + x \doteq x) \vdash \forall x.\mathbf{0} + x \doteq x}
}{T'_A, \mathbf{0} + \mathbf{0} \doteq \mathbf{0} \to (\forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x)) \to (\forall x.\mathbf{0} + x \doteq x) \vdash \forall x.\mathbf{0} + x \doteq x}
$$

$T_A \models \forall x.\mathbf{0} + x \doteq x$

$$\text{(SUBST)}\ \dfrac{T'_A, \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq \mathbf{0} + s(x), \forall x.\mathbf{0} + x \doteq x}{T'_A, \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}$$

$$(\forall L)\ \dfrac{}{T'_A, \forall y.\mathbf{0} + s(y) \doteq s(\mathbf{0} + y), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq}$$

$$(\forall L)\ \dfrac{}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}$$

$$\text{(SUBST)}\ \dfrac{}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}$$

$$(\to R)\ \dfrac{}{T'_A \vdash \mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}$$

$$(\forall R)\ \dfrac{}{T'_A \vdash \forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}$$

$$\dfrac{\mathbf{0} + \mathbf{0} \doteq \mathbf{0}, \forall x.\mathbf{0} + x \doteq x}{\doteq \mathbf{0}, \forall x.\mathbf{0} + x \doteq x}$$

$$(\to L)\ \dfrac{}{T'_A, (\forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x)) \to (\forall x.\mathbf{0}}$$

$$T'_A, \mathbf{0} + \mathbf{0} \doteq \mathbf{0} \to (\forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x)) \to (\forall x.\mathbf{0} + x \doteq x) \vdash \forall x.\mathbf{0} + x \doteq x$$

$T_A \models \forall x.\mathbf{0} + x \doteq x$

$$\text{(SUBST)}\ \frac{T'_A, \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq \ldots}{T'_A, \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq \ldots}$$

$$(\forall \mathrm{L})\ \frac{T'_A, \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq \ldots}{T'_A, \forall y.\mathbf{0} + s(y) \doteq s(\mathbf{0} + y), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq \ldots}$$

$$(\forall \mathrm{L})\ \frac{T'_A, \forall y.\mathbf{0} + s(y) \doteq s(\mathbf{0} + y), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq \ldots}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall \ldots}$$

$$\text{(SUBST)}\ \frac{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall \ldots}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(x), \forall x.\ldots}$$

$$(\rightarrow \mathrm{R})\ \frac{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(x), \forall x.\ldots}{T'_A \vdash \mathbf{0} + x \doteq x \rightarrow \mathbf{0} + s(x) \doteq s(x), \forall x \ldots}$$

$$(\forall \mathrm{R})\ \frac{T'_A \vdash \mathbf{0} + x \doteq x \rightarrow \mathbf{0} + s(x) \doteq s(x), \forall x \ldots}{T'_A \vdash \forall x.\mathbf{0} + x \doteq x \rightarrow \mathbf{0} + s(x) \doteq s(x), \ldots}$$

$$(\forall \mathrm{L})\ \frac{T'_A, \mathbf{0} + \mathbf{0} \doteq \mathbf{0} \vdash \mathbf{0} + \mathbf{0} \doteq \mathbf{0}, \forall x.\mathbf{0} + x \doteq x}{T'_A \vdash \mathbf{0} + \mathbf{0} \doteq \mathbf{0}, \forall x.\mathbf{0} + x \doteq x} \qquad (\rightarrow \mathrm{L})\ \frac{T'_A \vdash \forall x.\mathbf{0} + x \doteq x \rightarrow \mathbf{0} + s(x) \doteq s(x), \ldots}{T'_A, (\forall x.\mathbf{0} + x \doteq x \rightarrow \mathbf{0} + \ldots}$$

$$(\rightarrow \mathrm{L})\ \frac{T'_A \vdash \mathbf{0} + \mathbf{0} \doteq \mathbf{0}, \forall x.\mathbf{0} + x \doteq x \qquad T'_A, (\forall x.\mathbf{0} + x \doteq x \rightarrow \mathbf{0} + \ldots}{T'_A, \mathbf{0} + \mathbf{0} \doteq \mathbf{0} \rightarrow (\forall x.\mathbf{0} + x \doteq x \rightarrow \mathbf{0} + s(x) \doteq s(x)) \rightarrow (\forall x.\mathbf{0} + x \doteq \ldots}$$

$T_A \models \forall x.\mathbf{0} + x \doteq x$

$$\frac{\dfrac{\dfrac{T'_A, \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq \mathbf{0} + s(x), \forall x.\mathbf{0} + x \doteq x}{T'_A, \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}\,(\text{SUBST})}{T'_A, \forall y.\mathbf{0} + s(y) \doteq s(\mathbf{0} + y), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}\,(\forall\text{L})}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}\,(\forall\text{L})$$

$$\frac{\dfrac{\dfrac{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}\,(\text{SUBST})}{T'_A \vdash \mathbf{0} + x \doteq x \rightarrow \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}\,(\rightarrow\text{R})}{T'_A \vdash \forall x.\mathbf{0} + x \doteq x \rightarrow \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}\,(\forall\text{R})$$

$$\frac{\cdots\, x \doteq x \qquad\qquad T'_A \vdash \forall x.\mathbf{0} + x \doteq x \rightarrow \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x \qquad\qquad T'_A, \forall x.\mathbf{0} + \cdots}{T'_A, (\forall x.\mathbf{0} + x \doteq x \rightarrow \mathbf{0} + s(x) \doteq s(x)) \rightarrow (\forall x.\mathbf{0} + x \doteq x) \vdash \forall x.\mathbf{0} \cdots}\,(\rightarrow\text{L})$$

$$\frac{\cdots\, x \qquad T'_A, (\forall x.\mathbf{0} + x \doteq x \rightarrow \mathbf{0} + s(x) \doteq s(x)) \rightarrow (\forall x.\mathbf{0} + x \doteq x) \vdash \forall x.\mathbf{0} \cdots}{T'_A, \mathbf{0} + \mathbf{0} \doteq \mathbf{0} \rightarrow (\forall x.\mathbf{0} + x \doteq x \rightarrow \mathbf{0} + s(x) \doteq s(x)) \rightarrow (\forall x.\mathbf{0} + x \doteq x) \vdash \forall x.\mathbf{0} + x \doteq x}$$

$T_A \models \forall x.\mathbf{0} + x \doteq x$

$$) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq \mathbf{0} + s(x), \forall x.\mathbf{0} + x \doteq x$$

$$) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x$$

$$y) \doteq s(\mathbf{0} + y), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x$$

$$T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x$$

$$T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x$$

$$T'_A \vdash \mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x$$

$$'_A \vdash \forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x \qquad T'_A, \forall x.\mathbf{0} + x \doteq x \vdash \forall x.\mathbf{0} + x \doteq x$$

$$T'_A, (\forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x)) \to (\forall x.\mathbf{0} + x \doteq x) \vdash \forall x.\mathbf{0} + x \doteq x$$

$$x \to \mathbf{0} + s(x) \doteq s(x)) \to (\forall x.\mathbf{0} + x \doteq x) \vdash \forall x.\mathbf{0} + x \doteq x$$

A Closer Look

$$T'_A \vdash \forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x$$

A Closer Look

$$(\forall R)\ \frac{T'_A \vdash \mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}{T'_A \vdash \forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}$$

A Closer Look

$$
(\to\mathrm{R})\ \dfrac{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}{(\forall\mathrm{R})\ \dfrac{T'_A \vdash \mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}{T'_A \vdash \forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}}
$$

A Closer Look

$$
\dfrac{\dfrac{\dfrac{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x. \mathbf{0} + x \doteq x}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(x), \forall x. \mathbf{0} + x \doteq x}\;(\textsc{subst})}{T'_A \vdash \mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x. \mathbf{0} + x \doteq x}\;(\to\mathrm{R})}{T'_A \vdash \forall x. \mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x. \mathbf{0} + x \doteq x}\;(\forall\mathrm{R})
$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{T'_A, \forall y.\mathbf{0} + s(y) \doteq s(\mathbf{0} + y), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}\,(\forall \mathrm{L})}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}\,(\textsc{subst})}{T'_A \vdash \mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}\,(\to\mathrm{R})}{T'_A \vdash \forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}\,(\forall \mathrm{R})
$$

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\dfrac{\dfrac{T'_A, \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}{T'_A, \forall y.\mathbf{0} + s(y) \doteq s(\mathbf{0} + y), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}\;(\forall\mathrm{L})}{\dfrac{\dfrac{\dfrac{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}\;(\textsc{subst})}{T'_A \vdash \mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}\;(\to\mathrm{R})}{T'_A \vdash \forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}\;(\forall\mathrm{R})}\;(\forall\mathrm{L})
$$

$$
(\forall\mathrm{R})\;\dfrac{(\to\mathrm{R})\;\dfrac{(\textsc{subst})\;\dfrac{(\forall\mathrm{L})\;\dfrac{(\forall\mathrm{L})\;\dfrac{(\textsc{subst})\;\dfrac{T'_A, \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq \mathbf{0} + s(x), \forall x.\mathbf{0} + x \doteq x}{T'_A, \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}}{T'_A, \forall y.\mathbf{0} + s(y) \doteq s(\mathbf{0} + y), \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(\mathbf{0} + x), \forall x.\mathbf{0} + x \doteq x}}{T'_A, \mathbf{0} + x \doteq x \vdash \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}}{T'_A \vdash \mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}}{T'_A \vdash \forall x.\mathbf{0} + x \doteq x \to \mathbf{0} + s(x) \doteq s(x), \forall x.\mathbf{0} + x \doteq x}
$$

What Have We Proved?

- we have shown that $T'_A \vdash_{\mathrm{LK}} \forall x.\mathbf{0} + x \doteq x$
- so by soundness $T'_A \models \forall x.\mathbf{0} + x \doteq x$
- also $T_A \models \forall x.\mathbf{0} + x \doteq x$:
    - assume $\mathcal{M}, \sigma \models T_A$
    - then $\mathcal{M}, \sigma \models \varphi$ for every $\varphi \in T_A$
    - but $T'_A \subseteq T_A$, so $\mathcal{M}, \sigma \models \varphi$ for every $\varphi \in T'_A$
    - hence $\mathcal{M}, \sigma \models T'_A$
    - by the above, this means $\mathcal{M}, \sigma \models \forall x.\mathbf{0} + x \doteq x$

Proving Commutativity

- this is just the first step towards proving $T_A \models \forall x.\forall y.x + y \doteq y + x$
- the whole proof can be done in sequent calculus, but it is very long and tedious
- many other laws about $+$ and $\times$ can be proved as well

Beyond Addition and Multiplication

- T_A contains no axioms for exponentiation
- we could add them, yielding T_A^{exp}:
  - $\forall x. x^{\mathbf{0}} \doteq s(\mathbf{0})$
  - $\forall x. \forall y. x^{s(y)} \doteq x^y \times x$
- but we do not need to do that:

Expressing Exponentiation in T_A

For every formula φ using exponentiation, we can find a formula φ' not using exponentiation such that $T_A^{\mathrm{exp}} \models \varphi$ iff $T_A \models \varphi'$!

- in fact, we can express any computable function in T_A

Limits of First-order Logic

Compactness Theorem

Compactness Theorem

A set of formulas Γ is satisfiable iff every finite subset of Γ is satisfiable.

- left-to-right direction is easy:
- right-to-left direction is easy if Γ is finite:
- otherwise, this direction is quite hard to prove

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- left-to-right direction is easy:
  - assume Γ is satisfiable; then we have $\mathcal{M}, \sigma$ such that $\mathcal{M}, \sigma \models \gamma$ for every $\gamma \in \Gamma$
  - let $\Gamma' \subseteq \Gamma$; then for any $\gamma' \in \Gamma'$ we have $\gamma' \in \Gamma'$, so $\mathcal{M}, \sigma \models \gamma'$; hence $\mathcal{M}, \sigma \models \Gamma'$
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  Γ is a finite subset of itself
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To determine satisfiability of Γ, its infinite subsets are unimportant!

Cardinality Formulas

Assume $\mathcal{M} \models \varphi$ for any of the following formulas φ; what does this say about the domain D of $\mathcal{M}$?

- $\exists x.\exists y.\neg(x \doteq y)$
  D has at least two elements
- $\exists x.\exists y.\exists z.\neg(x \doteq y) \wedge \neg(x \doteq z) \wedge \neg(y \doteq z)$
  D has at least three elements
- $\forall x.\forall y.\forall z.x \doteq y \vee x \doteq z \vee y \doteq z$
  D has at most two elements
- $(\forall x.\forall y.f(x) \doteq f(y) \rightarrow x \doteq y) \wedge \neg(\forall z.\exists x.f(x) \doteq z)$
  D is infinite

Can you write a formula φ such that $\mathcal{M} \models \varphi$ iff D is finite?

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Defining Finiteness

- $\mathcal{M} \models (\forall x.\forall y.f(x) \doteq f(y) \rightarrow x \doteq y) \land \neg(\forall z.\exists x.f(x) \doteq z)$ iff domain is infinite:
  - $[\![f]\!]_{\mathcal{M}}$ is injective
  - $[\![f]\!]_{\mathcal{M}}$ is not surjective
- is it true that $\mathcal{M} \models \neg(\forall x.\forall y.f(x) \doteq f(y) \rightarrow x \doteq y) \lor (\forall z.\exists x.f(x) \doteq z)$ iff domain is finite?

No! This formula just says that $[\![f]\!]_{\mathcal{M}}$ is either not injective or surjective. We can choose domain $\mathbb{N}$ and $[\![f]\!]_{\mathcal{M}}(n) := n$.

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Surprise!

Finiteness Is Not First-order Definable

There is no first-order formula φ such that $\mathcal{M} \models \varphi$ iff the domain of $\mathcal{M}$ is finite.

- assume we had such a formula φ_f
- let Λ be the set of all λ_n
- consider finite subsets S_f of $\Lambda \cup \{\varphi_f\}$:
  - if $S_f \subseteq \Lambda$, it is satisfiable
  - otherwise $S_f = \Lambda' \cup \{\varphi_f\}$ for a finite $\Lambda' \subseteq \Lambda$, and it is still satisfiable
- so all finite subsets of $\Lambda \cup \{\varphi_f\}$ are satisfiable, but the set itself is not; this contradicts the Compactness Theorem
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- if $\mathcal{M} \models \Lambda$ then the domain of $\mathcal{M}$ must be infinite; so $\Lambda \cup \{\varphi_f\}$ is unsatisfiable
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Reflexive Transitive Closure

- assume our signature contains binary relation symbols r, s
- can you find a formula φ^* such that $\mathcal{M} \models \varphi$ iff $[\![s]\!]_{\mathcal{M}}$ is the reflexive transitive closure of $[\![r]\!]_{\mathcal{M}}$?
- no!
  - assume we had such a φ^*
  - define, for every $n \in \mathbb{N}$, a formula $r_n(x, y)$ with free variables x and y such that $\mathcal{M}, \sigma \models r_n(x, y)$ iff $\sigma(y)$ is reachable from $\sigma(x)$ through n iterations of $[\![r]\!]_{\mathcal{M}}$
    for example, $r_3(x, y) := \exists z_1.\exists z_2.r(x, z_1) \wedge r(z_1, z_2) \wedge r(z_2, y)$
  - define, for every $n \in \mathbb{N}$, a formula $\delta_n := s(x, y) \wedge \neg r_n(x, y)$
  - let Δ be set of all these formulas
  - then $\Delta \cup \{\varphi^*\}$ is unsatisfiable, but every finite subset is satisfiable
  - contradiction: φ^* cannot exist!

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  - define, for every $n \in \mathbb{N}$, a formula $r_n(x, y)$ with free variables x and y such that $\mathcal{M}, \sigma \models r_n(x, y)$ iff $\sigma(y)$ is reachable from $\sigma(x)$ through n iterations of $[\![r]\!]_{\mathcal{M}}$
    for example, $r_3(x, y) := \exists z_1.\exists z_2.r(x, z_1) \wedge r(z_1, z_2) \wedge r(z_2, y)$
  - define, for every $n \in \mathbb{N}$, a formula $\delta_n := s(x, y) \wedge \neg r_n(x, y)$
  - let Δ be set of all these formulas
  - then $\Delta \cup \{\varphi^*\}$ is unsatisfiable, but every finite subset is satisfiable
  - contradiction: φ^* cannot exist!

Fixpoints

- one might think that the following formula should do the trick:

$$\varphi^* := \forall x.\forall y.s(x,y) \leftrightarrow x \doteq y \vee (\exists z.r(x,z) \wedge s(z,y))$$

- but consider the structure $\mathcal{M} = \langle \mathbb{N}, \langle [\![]\!]_{\mathtt{F}}, [\![]\!]_{\mathcal{R}} \rangle\rangle$ with $[\![r]\!]_{\mathcal{R}} := \{(n,n) \mid n \in \mathbb{N}\}$ and $[\![s]\!]_{\mathcal{R}} := \mathbb{N} \times \mathbb{N}$
- then $\mathcal{M} \models \varphi^*$, but $[\![s]\!]_{\mathcal{R}}$ is not the reflexive transitive closure of $[\![r]\!]_{\mathcal{R}}$
- roughly, the reflexive transitive closure of r is the *least* fixpoint of the function

$$F(s) := \{(x,y) \mid x \doteq y \vee \exists z.r(x,z) \wedge s(z,y)\}$$

but φ^* only ensures that s is *some* fixpoint of F

Conclusion

- first-order logic is enough to formalise arithmetic
- large parts of mathematics can be done in first-order logic
- sequent calculus is a sound and complete deductive system for first-order logic
- for analysis, however, we need *second-order logic*
- in second-order logic we can quantify over propositions, so we can write the induction axiom as a single formula:

$$\forall P . P(\mathbf{0}) \to (\forall x . P(x) \to P(s(x))) \to (\forall x . P(x))$$

- but there are no complete deductive systems for second-order logic (Gödel's Incompleteness Theorem)