Some More Logic Puzzles (Solution)

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June 29, 2010

Assume A says any of the following things; what can you deduce about A and B?

- If I am a knight, then so is B.

$$A \leftrightarrow \underline{A \rightarrow B}$$
$$\Leftrightarrow \quad \{ \text{ Unfolding } \rightarrow \}$$
$$\underline{A \leftrightarrow B \leftrightarrow A \vee B}$$
$$\Leftrightarrow \quad \{ \text{ Golden Rule } \}$$
$$A \wedge B$$

  So A and B are both knights.

- If B is a knight, then so am I.

$$A \leftrightarrow \underline{B \rightarrow A}$$
$$\Leftrightarrow \quad \{ \text{ Unfolding } \rightarrow \}$$
$$\underline{A \leftrightarrow A} \leftrightarrow B \vee A$$
$$\Leftrightarrow \quad \{ \text{ Unfolding } \top \text{ twice } \}$$
$$B \vee A$$

  So at least one of A and B are knights.

- If I am a knave, then B is a knight.
  We first prove an auxiliary law:

$$\underline{P} \leftrightarrow P \vee Q$$
$$\Leftrightarrow \quad \{ P \vee \bot \leftrightarrow P \}$$
$$\underline{P \vee \bot \leftrightarrow P \vee Q}$$
$$\Leftrightarrow \quad \{ \text{ Distributivity of } \vee \}$$
$$P \vee \underline{(\bot \leftrightarrow Q)}$$
$$\Leftrightarrow \quad \{ \text{ Symmetry of } \leftrightarrow \text{ and unfolding } \neg \}$$
$$P \vee \neg Q$$

Now we work on the main problem:

$$A \leftrightarrow \underline{\neg A \rightarrow B}$$
$$\Leftrightarrow \qquad \{ \text{ Unfolding } \rightarrow \}$$
$$A \leftrightarrow B \leftrightarrow \underline{\neg A \vee B}$$
$$\Leftrightarrow \qquad \{ \text{ Symmetry of } \vee \}$$
$$A \leftrightarrow B \leftrightarrow \underline{B \vee \neg A}$$
$$\Leftrightarrow \qquad \{ \ P \leftrightarrow P \vee Q \leftrightarrow P \vee \neg Q \ \}$$
$$A \leftrightarrow \underline{B \leftrightarrow B} \leftrightarrow B \vee A$$
$$\Leftrightarrow \qquad \{ \text{ Unfolding } \top \text{ twice } \}$$
$$A \leftrightarrow \underline{B \vee A}$$
$$\Leftrightarrow \qquad \{ \text{ Symmetry of } \vee \}$$
$$\underline{A \leftrightarrow A \vee B}$$
$$\Leftrightarrow \qquad \{ \ P \leftrightarrow P \vee Q \leftrightarrow P \vee \neg Q \ \}$$
$$A \vee \neg B$$

So either A is a knight, or B is a knave, or both.

- If I am a knight, then B is a knave.

$$A \leftrightarrow \underline{A \rightarrow \neg B}$$
$$\Leftrightarrow \qquad \{ \text{ Unfolding } \rightarrow \}$$
$$\underline{A \leftrightarrow \neg B \leftrightarrow A \vee \neg B}$$
$$\Leftrightarrow \qquad \{ \text{ Golden Rule } \}$$
$$A \wedge \neg B$$

So A is a knight and B is a knave.

- If B is a knave, then I am a knave.

  Again, we first establish an auxiliary result.

$$\underline{\neg Q \rightarrow \neg P}$$
$$\Leftrightarrow \qquad \{ \text{ Unfolding } \rightarrow \}$$
$$\neg P \leftrightarrow \underline{\neg Q} \vee \neg P$$
$$\Leftrightarrow \qquad \{ \text{ Unfolding } \neg \}$$
$$\neg P \leftrightarrow \underline{(Q \leftrightarrow \bot) \vee \neg P}$$
$$\Leftrightarrow \qquad \{ \text{ Distributivity of } \vee \}$$
$$\neg P \leftrightarrow Q \vee \neg P \leftrightarrow \underline{\bot \vee \neg P}$$
$$\Leftrightarrow \qquad \{ \ P \vee \bot \leftrightarrow P \ \}$$

$$\neg P \leftrightarrow Q \vee \neg P \leftrightarrow \neg P$$
$\Leftrightarrow$ { Symmetry of $\leftrightarrow$ }
$$\underline{\neg P \leftrightarrow \neg P} \leftrightarrow Q \vee \neg P$$
$\Leftrightarrow$ { Unfolding $\top$ twice }
$$\underline{Q \vee \neg P}$$
$\Leftrightarrow$ { $P \leftrightarrow P \vee Q \leftrightarrow P \vee \neg Q$ }
$$Q \leftrightarrow \underline{Q \vee P}$$
$\Leftrightarrow$ { Symmetry of $\vee$ }
$$Q \leftrightarrow \underline{P \vee Q}$$
$\Leftrightarrow$ { Unfolding $\rightarrow$ }
$$P \rightarrow Q$$

Now the main problem is easily simplified:

$$A \leftrightarrow \underline{\neg B \rightarrow \neg A}$$
$\Leftrightarrow$ { $\neg Q \rightarrow \neg P \leftrightarrow P \rightarrow Q$ }
$$\underline{A \leftrightarrow A \rightarrow B}$$
$\Leftrightarrow$ { see first problem above }
$$A \wedge B$$

So both A and B are knights.

- B says one of us is a knight.

$$\underline{A \leftrightarrow B \leftrightarrow A \vee B}$$
$\Leftrightarrow$ { Golden Rule }
$$A \wedge B$$

So both A and B are knights.