Homework Assignment 2 Solution

[Compiled on July 1, 2009]

Problem 1

1. $(p \wedge q) \supset p$ PL
2. $\Box((p \wedge q) \supset p)$ Gen
3. $\Box(p \supset q) \supset (\Box p \subset \Box q)$ K
4. $\Box((p \wedge q) \supset p) \subset (\Box(p \wedge q) \supset p)$ 3.$[(p \wedge q)/p, p/q]$
5. $\Box(p \wedge q) \supset \Box p$ 2, 4, MP
6. $(p \wedge q) \supset q$ PL
7. $\Box((p \wedge q) \supset q)$ Gen
8. $\Box((p \wedge q) \supset q) \subset (\Box(p \wedge q) \supset q)$ 4.$[(p \wedge q)/p, q/q]$
9. $\Box(p \wedge q) \supset \Box q$ 7, 8, MP
10. $(p \supset q) \supset ((p \supset r) \supset (p \supset (q \wedge r)))$ PL
11. $(\Box(p \wedge q) \supset \Box p) \supset ((\Box(p \wedge q) \supset \Box q) \supset (\Box(p \wedge q) \supset (\Box p \wedge \Box q)))$ 10.$[\Box(p \wedge q)/p, \Box p/q, \Box q/r]$
12. $(\Box(p \wedge q) \supset \Box q) \supset (\Box(p \wedge q) \supset (\Box p \wedge \Box q))$ 5, 11, MP
13. $\Box(p \wedge q) \supset (\Box p \wedge \Box q)$ 9, 12, MP

Problem 2

1. $\varphi \supset \Box\Diamond\varphi$ B
2. $(p \supset q) \supset ((q \supset r) \supset (p \supset r))$ PL
3. $(\varphi \supset \Box\Diamond\varphi) \supset ((\Box\Diamond\varphi \supset \Box\psi) \supset (\varphi \supset \Box\psi))$ 2.$[\varphi/p, \Box\Diamond\varphi/q, \Box\psi/r]$
4. $(\Box\Diamond\varphi \supset \Box\psi) \supset (\varphi \supset \Box\psi)$ 1, 3, MP

5. $\Box(p \supset q) \supset (\Box p \supset \Box q)$ K

6. $\Box(\Diamond\varphi \supset \psi) \supset (\Box\Diamond\varphi \supset \Box\psi)$ 5.$[\Diamond\varphi/p,\psi/q]$

7. $(\Box(\Diamond\varphi \supset \psi) \supset (\Box\Diamond\varphi \supset \Box\psi)) \supset (((\Box\Diamond\varphi \supset \Box\psi) \supset (\varphi \supset \Box\psi)) \supset (\Box(\Diamond\varphi \supset \psi) \supset (\varphi \supset \Box\psi)))$ 2.$[\Box(\Diamond\varphi \supset \psi)/p,\ \Box\Diamond\varphi \supset \Box\psi/q,\ \varphi \supset \Box\psi/r]$

8. $((\Box\Diamond\varphi \supset \Box\psi) \supset (\varphi \supset \Box\psi)) \supset (\Box(\Diamond\varphi \supset \psi) \supset (\varphi \supset \Box\psi)))$ 6, 7, MP

9. $(\Box(\Diamond\varphi \supset \psi) \supset (\varphi \supset \Box\psi)))$ 4, 8, MP

Problem 3

We shall prove that $\Diamond\varphi \supset \Box\Diamond\varphi$ is valid in every model $\mathfrak{M} = (W, R, \pi)$ which is Euclidean.

For every $w \in W$, suppose $\mathfrak{M}, w \Vdash \Diamond\varphi$, then there exists $u \in W$ such that $(w, u) \in R$ and $\mathfrak{M}, u \Vdash \varphi$. Then for every $v \in W$ such that $(w, v) \in R$, by Euclidean, $(v, u) \in R$, i.e. $\mathfrak{M}, v \Vdash \Diamond\varphi$.

This means $\mathfrak{M}, w \Vdash \Box\Diamond\varphi$, and $\Diamond\varphi \supset \Box\Diamond\varphi$ is valid in $\mathfrak{M}$.

Problem 4

Let $\mathfrak{M} = (W, R, \pi)$. Suppose $w, u \in W$, $w \neq u$, $(w, u) \in R$, by canonical,

$$\{\varphi \mid \Box\varphi \in \Sigma_w\} \subseteq \Sigma_u$$

Since the system contains B, $\varphi \supset \Box\Diamond\varphi \in \Sigma_w$. For every $\varphi \in \Sigma_w$,

1. $\varphi \in \Sigma_w$
2. $\Box\Diamond\varphi \in \Sigma_w$ by B
3. $\Diamond\varphi \in \Sigma_u$ by canonical
4. $\{\Diamond\varphi \mid \varphi \in \Sigma_w\} \subseteq \Sigma_u$ by 1, 3
5. $\{\varphi \mid \Box\varphi \in \Sigma_u\} \subseteq \Sigma_w$ by Maximality II.1
6. $(u, w) \in R$ by canonical

So that $\mathfrak{M}$ is a symmetric model.