Elementary Logic

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Outline

The Language

- The following symbols are used in sentential logic

Symbol	Name	Remark
(	left parenthesis	punctuation
)	right parenthesis	punctuation
$\neg$	negation symbol	not
$\wedge$	conjunction symbol	and
$\vee$	disjuction symbol	or (inclusive)
$\rightarrow$	condition symbol	if __, then __
$\leftrightarrow$	biconditional symbol	if and only if
A_1	first sentence symbol	
A_2	second sentence symbol	
...		
A_n	nth sentence symbol	
...		

- The set of sentence symbols will be denoted by $\mathscr{S}$

Well-Formed Formulae (wff's)

- A set S of expressions is inductive if it has the following properties.
- A well-formed formula (wff) is defined as follows:
  - every sentence symbol is a wff;
  - if expressions α and β are wff's, then so are $(\neg\alpha)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, and $(\alpha \leftrightarrow \beta)$.
- The set of wffs generated from $\mathscr{S}$ is denoted by $\overline{\mathscr{S}}$

Truth Assignments

- Fix a set $\{T, F\}$ of truth values
- A truth assignment is a function

$$\nu : \mathscr{S} \to \{T, F\}$$

Extended Truth Assignment

- Define the extension $\overline{\nu} : \overline{\mathscr{S}} \to \{T, F\}$ by

$$\begin{array}{rcl}
\overline{\nu}(A) & = & \nu(A) \\
\overline{\nu}((\neg\alpha)) & = & \begin{cases} T & \text{if } \overline{\nu}(\alpha) = F \\ F & \text{otherwise} \end{cases} \\
\overline{\nu}((\alpha \wedge \beta)) & = & \begin{cases} T & \text{if } \overline{\nu}(\alpha) = T \text{ and } \overline{\nu}(\beta) = T \\ F & \text{otherwise} \end{cases} \\
\overline{\nu}((\alpha \vee \beta)) & = & \begin{cases} T & \text{if } \overline{\nu}(\alpha) = T \text{ or } \overline{\nu}(\beta) = T \\ F & \text{otherwise} \end{cases} \\
\overline{\nu}((\alpha \to \beta)) & = & \begin{cases} F & \text{if } \overline{\nu}(\alpha) = T \text{ and } \overline{\nu}(\beta) = F \\ T & \text{otherwise} \end{cases} \\
\overline{\nu}((\alpha \leftrightarrow \beta)) & = & \begin{cases} T & \text{if } \overline{\nu}(\alpha) = \overline{\nu}(\beta) \\ F & \text{otherwise} \end{cases}
\end{array}$$

Tautology

- A truth assignment ν satisfies a wff ϕ if $\overline{\nu}(\phi) = T$
- Let Σ be a set of wffs and ϕ a wff. Σ tautologically implies ϕ ($\Sigma \vDash \phi$) if every truth assignment satisfies every member of Σ also satisfies ϕ
- ϕ is a tautology if $\varnothing \vDash \phi$
- If $\sigma \vDash \tau$ and $\tau \vDash \sigma$, we say σ and τ are tautologically equivalent ($\sigma \vDash\dashv \tau$)
  - $\sigma \vDash \tau$ stands for $\{\sigma\} \vDash \tau$

Omitting Parentheses

To reduce the number of parentheses, we use the following convention:

- The outmost parentheses need not be explicitly mentioned. "$A \wedge B$" means $(A \wedge B)$
- The negation symbol applies to as little as possible. "$\neg A \wedge B$" means $(\neg A) \wedge B$
- The conjunction and disjunction symbols also apply to as little as possible. "$A \wedge B \to \neg C \vee D$" means $(A \wedge B) \to ((\neg C) \vee D)$
- Where one connective symbol is used repeatedly, grouping to the right. "$A \to B \to C$" means $A \to (B \to C)$

Boolean Functions

- A k-place Boolean function is a function from $\{T, F\}^k$ into $\{T, F\}$
- Suppose a wff α has sentence symbols among $A_1, \ldots, A_n$. The Boolean function B^n_α realized by α is defined by

$$B^n_\alpha(X_1, \ldots, X_n) = \overline{\nu}(\alpha)$$

where $\nu(A_i) = X_i \in \{T, F\}$ for each $i = 1, \ldots, n$

Facts about B^n_α

Theorem

Let α and β be wffs whose sentence symbols are among $A_1, \ldots, A_n$.

1. $\alpha \vDash \beta$ *iff for all* $\vec{X} \in \{T, F\}^n$, $B^n_\alpha(\vec{X}) = T$ *implies* $B^n_\beta(\vec{X}) = T$
2. $\alpha \vDash\Dashv \beta$ *iff* $B^n_\alpha = B^n_\beta$
3. $\vDash \alpha$ *iff ran* $B^n_\alpha = \{T\}$

Proof.

Observe that $\alpha \vDash \beta$ iff for all 2^n truth assignments ν, $\overline{\nu}(\alpha) = T$ implies $\overline{\nu}(\beta) = T$. □

Completeness of Connectives

Theorem

Let G be an n-place Boolean function with $n \geq 1$. There is a wff α such that $G = B^n_\alpha$

Proof.

If ran $G = \{F\}$, let $\alpha = A_1 \wedge \neg A_1$.
Otherwise, let G have the value T at $\vec{X_i} = \langle X_{i1}, X_{i2}, \ldots, X_{in} \rangle$ for $i = 1, \ldots, k$. Define

$$\begin{aligned}
\beta_{ij} &= \begin{cases} A_j & \text{if } X_{ij} = T \\ \neg A_j & \text{if } X_{ij} = F \end{cases} \\
\gamma_i &= \beta_{i1} \wedge \cdots \wedge \beta_{in} \\
\alpha &= \gamma_1 \vee \cdots \vee \gamma_k
\end{aligned}$$

It is straightforward to show $G = B^n_\alpha$ □

Disjunctive Normal Form

- A literal is either a sentence symbol A or its negation $\neg A$
- A wff α is in disjunctive normal form if

$$\alpha = \gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_k$$

where

$$\gamma_i = \beta_{i1} \wedge \beta_{i2} \wedge \cdots \beta_{in_i}$$

and β_{ij} is a literal

Corollary

For any wff ϕ, there is a tautologically equivalent wff α in disjunctive normal form

Compactness

- A set Σ of wffs is satisfiable if there is a truth assignment which satisfies every member of Σ
- Σ is finitely satisfiable if every finite subset of Σ is satisfiable
- In mathematics, compactness relates finite and infinite features
  - A set is compact if any open cover has a finite subcover
    - ★ bounded closed sets are compact; bounded open sets are not.

Proof of Compactness

Theorem

A set Σ of wffs is satisfiable iff it is finitely satisfiable

Proof.

Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of wffs. Define

$$\begin{aligned}\Delta_0 &= \Sigma \\ \Delta_{n+1} &= \begin{cases} \Delta_n \cup \{\alpha_{n+1}\} & \text{if this is finitely satisfiable} \\ \Delta_n \cup \{\neg\alpha_{n+1}\} & \text{otherwise}\end{cases}\end{aligned}$$

Let $\Delta = \cup_n \Delta_n$. Then (1) $\Sigma \subseteq \Delta$; (2) for any wff α, either $\alpha \in \Delta$ or $\neg\alpha \in \Delta$; and (3) Δ is finitely satisfiable.
Define a truth assignment ν by $\nu(A) = \mathrm{T}$ if $A \in \Delta$ for every sentence symbol A. Then ν satisfies ϕ iff $\phi \in \Delta$. Since $\Sigma \subseteq \Delta$, ν satisfies every member of Σ. □

Applications of Compactness

Corollary

If $\Sigma \vDash \tau$, there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vDash \tau$

Proof.

Suppose $\Sigma_0 \nvDash \tau$ for every finite $\Sigma_0 \subseteq \Sigma$. Then $\Sigma_0 \cup \{\tau\}$ is not satisfiable for any finite $\Sigma_0 \subseteq \Sigma$. Hence $\Sigma \cup \{\tau\}$ is not finitely satisfiable. Thus $\Sigma \cup \{\tau\}$ is not satisfiable. Therefore $\Sigma \nvDash \tau$. □

The Language

- Logical symbols
  - parentheses: (,)
  - sentential connectives: $\to$, $\neg$
  - variables: v_1, v_2, ...
  - equality symbol (optional): $\approx$
- Parameters
  - quantifier symbol: $\forall$
  - predicate symbols: n-place predicate symbols
  - constant symbols (or 0-place function symbols)
  - function symbols: n-place function symbols

Examples of First-Order Language

- Pure predicate language
  - equality: no
  - n-place predicate symbols: A_1^n, A_2^n, ...
  - constant symbols: a_1, a_2, ...
  - n-place function symbols ($n > 0$): none
- Language of set theory
  - equality: yes
  - predicate parameters: $\in$
  - constant symbols: $\varnothing$ (sometimes)
  - function symbols: none
- Language of elementary number theory
  - equality: yes
  - predicate parameters: $<$
  - constant symbols: 0
  - 1-place function symbols: S
  - 2-place function symbols: $+$, $\times$, and E

Examples

- "There is no set of which every set is a member."

$$\neg(\neg\forall v_1(\neg\forall v_2 \in v_2 v_1))$$
$$\text{or } \neg(\exists v_1(\forall v_2 \in v_2 v_1))$$

- "For any two sets, there is a set whose members are exactly the two given sets."

$$\forall v_1 v_2 \exists v_3 \forall v_4(\in v_4 v_3 \leftrightarrow \approx v_4 v_1 \vee \approx v_4 v_2)$$

- "Any nonzero natural number is the successor of some number."

$$\forall v_1(\neg \approx v_1 \mathbf{0} \to \exists v_2 \approx v_1 \mathbf{S} v_2)$$

Terms

- Terms are generated by variables, constant symbols, and function symbols
- Examples:

$$\begin{array}{rll} +v_2 S0 & \text{informally,} & v_2 + 1 \\ SSSS0 & \text{informally,} & 4 \\ +Ev_1 SS0 Ev_2 SSS0 & \text{informally,} & v_1^2 + v_2^3 \end{array}$$

Atomic Formulae

- An atomic formula is an expression of the form

$$Pt_1 \cdots t_n$$

  where P is an n-place predicate symbol (or equality), and $t_1, \ldots, t_n$ are terms
- Examples:

$$\begin{array}{rll} \approx v_1 S0 & \text{informally,} & v_1 = 1 \\ \in v_2 v_3 & \text{informally,} & v_2 \in v_3 \end{array}$$

Well-Formed Formulae

- The set of well-formed formulae (wff, or formulae) is generated from the atomic formulae by connective symbols ($\neg, \to$) and the quantifier symbol ($\forall$)
  - $\neg\gamma$, $\gamma \to \delta$, $\forall v_i \gamma$ are wffs provided γ, δ are
- Example:

$$\forall v_1((\neg \forall v_3(\neg \in v_3 v_1)) \to (\neg \forall v_2(\in v_2 v_1) \to (\neg \forall v_4(\in v_4 v_2 \to (\neg \in v_4 v_1)))))$$

informally

$$\forall v_1((\exists v_3 v_3 \in v_1) \to (\neg \forall v_2 v_2 \in v_1 \to (\neg \forall v_4 v_4 \in v_2 \to v_4 \notin v_1)))$$

- Nonexample: $\neg v_5$

Free Variables

- Let x be a variable and α a wff
- We say x occurs free in α if
  - x is a symbol in α when α is atomic
  - x occurs free in β when α is $\neg\beta$
  - x occurs free in β or in γ when α is $\beta \to \gamma$
  - x occurs free in β and $x \neq v_i$ when α is $\forall v_i \beta$
- If no variable occurs free in the wff α, we say α is a sentence
- Examples:
  - $\forall v_2(A v_2 \to B v_2)$ and $\forall v_3(P v_3 \to \forall v_3 Q v_3)$ are sentences
  - v_1 occurs free in $(\forall v_1 A v_1) \to B v_1$

Abbreviations

- Let α and β be formulae and x a variable
- $(\alpha \lor \beta)$ abbreviates $((\neg\alpha) \to \beta)$
- $(\alpha \land \beta)$ abbreviates $(\neg(\alpha \to (\neg\beta)))$
- $(\alpha \leftrightarrow \beta)$ abbreviates $((\alpha \to \beta) \land (\beta \to \alpha))$; that is,

$$(\neg((\alpha \to \beta) \to (\neg(\beta \to \alpha))))$$

- $\exists x \alpha$ abbreviates $(\neg \forall x(\neg\alpha))$
- $u \approx t$ abbreviates $\approx ut$ (and similarly for other 2-place predicate symbols)
- $u \not\approx t$ abbreviates $(\neg \approx ut)$ (and similarly for other 2-place predicate symbols)

Precedences

- Outermost parentheses may be dropped.
  - $\forall x \alpha \to \beta$ is $(\forall x \alpha \to \beta)$
- $\neg$, $\forall$, and $\exists$ apply to as little as possible.
  - $\neg\alpha \land \beta$ is $((\neg\alpha) \land \beta)$
  - $\forall x \alpha \to \beta$ is $((\forall x \alpha) \to \beta)$
- $\land$ and $\lor$ apply to as little as possible, subject to above
  - $\neg\alpha \land \beta \to \gamma$ is $(((\neg\alpha) \land \beta) \to \gamma)$
- When connective is used repeatedly, group them to the right
  - $\alpha \to \beta \to \gamma$ is $\alpha \to (\beta \to \gamma)$

Notation Conventions

- Predicate symbols: A, B, C, etc. Also $\in$, $<$
- Variables: v_i, u, x, y, etc.
- Function symbols: f, g, h, etc. Also S, $+$, etc.
- Constant symbols: a, b, c, etc. Also 0
- Terms: u, t
- Formulae: α, β, γ, etc.
- Sentences: σ, τ, etc.
- Set of formulae: Σ, Δ, Γ, etc.
- Structures: $\mathfrak{U}$, $\mathfrak{B}$, etc.

Structures

- A structure $\mathfrak{U}$ for a first-order language is a function whose domain is the set of parameters such that
  1. $\mathfrak{U}$ assigns to $\forall$ a nonempty set $|\mathfrak{U}|$, called the universe of $\mathfrak{U}$
  2. $\mathfrak{U}$ assigns to each n-place predicate symbol P an n-ary relation $P^{\mathfrak{U}} \subseteq |\mathfrak{U}|^n$
  3. $\mathfrak{U}$ assigns to each constant symbol c a member $c^{\mathfrak{U}} \in |\mathfrak{U}|$
  4. $\mathfrak{U}$ assigns to each n-place function symbol f an n-ary function $f^{\mathfrak{U}} : |\mathfrak{U}|^n \to |\mathfrak{U}|$
- Note that $|\mathfrak{U}|$ is nonempty and $f^{\mathfrak{U}}$ is not a partially-defined function

Examples of Structures

- In the language for set theory. Define
  - $|\mathfrak{U}| =$ the set of natural numbers
  - $\in^{\mathfrak{U}} = \{\langle m, n\rangle : m < n\}$
- Consider $\exists x \forall y \neg y \in x$
  - there is a natural number such that no natural number is smaller
- Informally, we would like to say $\exists x \forall y \neg y \in x$ is true in $\mathfrak{U}$ or $\mathfrak{U}$ is a model of the sentence

Satisfaction $\vDash_{\mathfrak{U}} \phi[s]$ I

Let ϕ be a wff, $\mathfrak{U}$ a structure, and $s : V \to |\mathfrak{U}|$ from the set V of variables to the universe of $\mathfrak{U}$

- *Terms*. Define the extension $\overline{s} : T \to |\mathfrak{U}|$ from terms to the universe by
  1. for variable x, $\overline{s}(x) = s(x)$
  2. for constant symbol c, $\overline{s}(c) = c^{\mathfrak{U}}$
  3. if $t_1, \ldots, t_n$ are terms and f is an n-place function symbol, $\overline{s}(ft_1 \cdots t_n) = f^{\mathfrak{U}}(\overline{s}(t_1), \ldots, \overline{s}(t_n))$
- *Atomic formulae*. Define
  1. $\vDash_{\mathfrak{U}} \approx t_1 t_2[s]$ if $\overline{s}(t_1) = \overline{s}(t_2)$
  2. for n-place predicate parameter P, $\vDash_{\mathfrak{U}} Pt_1 \cdots t_n[s]$ if $\langle \overline{s}(t_1), \ldots, \overline{s}(t_n)\rangle \in P^{\mathfrak{U}}$

Satisfaction $\models_{\mathfrak{A}} \phi[s]$ II

- *Other wffs*. Define
  1. $\models_{\mathfrak{A}} \neg\phi[s]$ if $\not\models_{\mathfrak{A}} \phi[s]$
  2. $\models_{\mathfrak{A}} (\phi \rightarrow \psi)[s]$ if $\not\models_{\mathfrak{A}} \phi[s]$ or $\models_{\mathfrak{A}} \psi[s]$
  3. $\models_{\mathfrak{A}} \forall x\phi[s]$ if for every $d \in |\mathfrak{A}|$, we have $\models_{\mathfrak{A}} \phi[s(x|d)]$ where

$$s(x|d)(y) = \begin{cases} s(y) & \text{if } y \neq x \\ d & \text{if } y = x \end{cases}$$

Logical Implication

Definition

Let Γ be a set of wffs, ϕ a wff. Γ logically implies ϕ ($\Gamma \models \phi$) if for every structure $\mathfrak{A}$ and every function $s : V \rightarrow |\mathfrak{A}|$ such that $\mathfrak{A}$ satisfies every member of Γ with s, $\mathfrak{A}$ also satisfies ϕ with s

- ϕ and ψ are logically equivalent ($\phi \models\!\!\!\dashv \psi$) if $\phi \models \psi$ and $\psi \models \phi$
- A wff ϕ is valid if $\varnothing \models \phi$ (or just $\models \phi$)

Relevant Valuation

Theorem

Assume $s_1, s_2 : V \rightarrow |\mathfrak{A}|$ such that s_1 and s_2 agree at all variables occurring free in ϕ. Then $\models_{\mathfrak{A}} \phi[s_1]$ iff $\models_{\mathfrak{A}} \psi[s_2]$.

Proof.

By induction.

- $\phi = Pt_1 \cdots t_n$. Observe $\overline{s_1}(t) = \overline{s_2}(t)$ for any term t occurring in ϕ (why?)
- $\phi = \neg\alpha$ or $\alpha \rightarrow \beta$. By inductive hypothesis
- $\phi = \forall x\psi$. Then free variables in ϕ are free variables in ψ except x. Thus $s_1(x|d)$ and $s_2(x|d)$ agree at free variables in ψ for any $d \in |\mathfrak{A}|$. By inductive hypothesis, $\models_{\mathfrak{A}} \psi[s_1(x|d)]$ iff $\models_{\mathfrak{A}} \psi[s_2(x|d)]$ for any $d \in |\mathfrak{A}|$.

$\square$

Truth and Models

Corollary

For a sentence σ, either

(a) *$\mathfrak{A}$ satisfies σ with every function s; or*

(b) *$\mathfrak{A}$ does not satisfy σ with any such function*

- If (a) holds, we say σ is true in $\mathfrak{A}$ or $\mathfrak{A}$ is a model of σ
- If (b) holds, we say σ is false in $\mathfrak{A}$
- $\mathfrak{A}$ is a model of a set Σ of sentences iff it is a model of every member of Σ

Corollary

For a set $\Sigma; \tau$ of sentences. $\Sigma \models \tau$ iff every model of Σ is a model of τ

Logical and Tautological Implications

- Consider the problem of determining $\vDash \phi$ when
  - ϕ is in sentential logic; and
  - ϕ is in first-order logic
- For sentential logic, there is an effective procedure
  - by truth table
- For first-order logic, we have to consider all structures
  - there are infinitely many structures!
  - the validity problem is in fact undecidable

Notational Convention

- By our notational convention, the following statements can be proved
  - $\vDash_{\mathfrak{A}} (\alpha \wedge \beta)[s]$ iff $\vDash_{\mathfrak{A}} \alpha[s]$ and $\vDash_{\mathfrak{A}} \beta[s]$; similarly for $\vee$ and $\leftrightarrow$
  - $\vDash_{\mathfrak{A}} \exists x \alpha[s]$ iff there is some $d \in |\mathfrak{A}|$ such that $\vDash_{\mathfrak{A}} \alpha[s(x|d)]$

Definability of Structures

- Let Σ be a set of sentences. $\mathrm{Mod}(\Sigma)$ denotes the class of all models of Σ. That is

$$\mathrm{Mod}(\Sigma) \quad = \quad \{\mathfrak{A} : \vDash_{\mathfrak{A}} \sigma \text{ for all } \sigma \in \Sigma\}$$

- A class $\mathscr{K}$ of structures is an elementary class (EC) if $\mathscr{K} = \mathrm{Mod}(\tau)$ for some sentence τ. $\mathscr{K}$ is an elementary class in the wider sense (EC_Δ) if $\mathscr{K} = \mathrm{Mod}(\Sigma)$ for some set Σ of sentences

Examples

- A structure (A, R) with $R \subseteq A \times A$ is an ordered set if R is transitive and satisfies trichotomy condition
  - that is, exactly one of $\langle a, b\rangle \in R$, $a = b$, $\langle b, a\rangle \in R$ holds
- The class of nonempty ordered sets is an elementary class

$$\begin{aligned} \tau \quad = \quad & \forall x \forall y \forall z (xRy \to yRz \to xRz) \wedge \\ & \forall x \forall y (xRy \vee x \approx y \vee yRx) \wedge \\ & \forall x \forall y (xRy \to \neg yRx) \end{aligned}$$

- The class of infinite sets is EC_Δ

$$\begin{aligned} \lambda_2 \quad &= \quad \exists x \exists y x \not\approx y \\ \lambda_3 \quad &= \quad \exists x \exists y \exists z (x \not\approx y \wedge x \not\approx z \wedge y \not\approx z) \\ & \quad\;\; \cdots \\ \Sigma \quad &= \quad \{\lambda_2, \lambda_3, \ldots,\} \end{aligned}$$

Definability within a Structure

- Fix a structure $\mathfrak{A}$
- Let ϕ be a formula with free variables $v_1, \ldots, v_k$
- For $a_1, \ldots, a_k \in |\mathfrak{A}|$, $\models_{\mathfrak{A}} \phi[\![a_1, \ldots, a_k]\!]$ means that $\mathfrak{A}$ satisfies ϕ with some $s : V \to |\mathfrak{A}|$ where $s(v_i) = a_i$ for $1 \le i \le k$
- The k-ary relation defined by ϕ is the relation
$$\{\langle a_1, \ldots, a_k\rangle : \models_{\mathfrak{A}} \phi[\![a_1, \ldots, a_k]\!]\}$$
- A k-ary relation on $|\mathfrak{A}|$ is definable if there is a formula defining it

Examples

- Consider the language of number theory with the intended structure $\mathfrak{N} = (\mathbb{N}, 0, S, +, -, \cdot)$
- The ordering relation $\{\langle m, n\rangle : m < n\}$ is defined by $\exists v_3 v_1 + S v_3 \approx v_2$
- For any $n \in \mathbb{N}$, $\{n\}$ is definable. For instance, $\{2\}$ is defined by $v_1 \approx SS0$
  - we hence say n is a definable element in $\mathfrak{N}$
- The set of primes is definable. Consider
$$\exists v_3 S0 + S v_3 \approx v_1 \wedge$$
$$\forall v_2 \forall v_3 (v_1 \approx v_2 \cdot v_3 \to v_2 \approx S0 \vee v_3 \approx S0)$$

Homomorphisms

- Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures. A mapping $h : |\mathfrak{A}| \to |\mathfrak{B}|$ is a homomorphism if
  - For each n-place predicate symbol P and n-tuple $\langle a_1, \ldots, a_n\rangle \in |\mathfrak{A}|^n$,
$$\langle a_1, \ldots, a_n\rangle \in P^{\mathfrak{A}} \text{ iff } \langle h(a_1), \ldots, h(a_n)\rangle \in P^{\mathfrak{B}}$$
  - For each n-place function symbol f and n-tuple $\langle a_1, \ldots, a_n\rangle \in |\mathfrak{A}|^n$,
$$h(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n))$$
- If h is one-to-one, it is called an isomorphism
- If there is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, we say $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic (in notation, $\mathfrak{A} \cong \mathfrak{B}$)

Examples

- Consider $(\mathbb{Z}^+, <^+_{\mathbb{Z}})$ and $(\mathbb{N}, <_{\mathbb{N}})$. The function $h(n) = n - 1$ is an isomorphism from $(\mathbb{Z}^+, <^+_{\mathbb{Z}})$ onto $(\mathbb{N}, <_{\mathbb{N}})$
- Consider two structures $\mathfrak{A}$ and $\mathfrak{B}$ with $|\mathfrak{A}| \subseteq |\mathfrak{B}|$. The identity map $(i(n) = n)$ is an isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ iff
  - $P^{\mathfrak{A}}$ is the restriction of $P^{\mathfrak{B}}$ to $|\mathfrak{A}|$ for every predicate symbol P; and
  - $f^{\mathfrak{A}}$ is the restriction of $f^{\mathfrak{B}}$ to $|\mathfrak{A}|$ for every function symbol f
- In this case, we say $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, and $\mathfrak{B}$ is an extension of $\mathfrak{A}$
- $(\mathbb{Z}^+, <^+_{\mathbb{Z}})$ is a substructure of $(\mathbb{N}, <_{\mathbb{N}})$

Homomorphism Theorem

Theorem

Let h be a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, and $s : V \to |\mathfrak{A}|$.

1. *For any term t, $h(\overline{s}(t)) = \overline{h \circ s}(t)$;*
2. *For any quantifier-free formula α without equality symbol, $\models_{\mathfrak{A}} \alpha[s]$ iff $\models_{\mathfrak{B}} a[h \circ s]$;*
3. *If h is one-to-one, then 2 holds even when α contains equality symbol;*
4. *If h is onto, then 2 holds even when α has quantifiers.*

Proof of Homomorphism Theorem I

1. By induction on t.
2. For atomic formula such as Pt, we have

$$\begin{aligned} \models_{\mathfrak{A}} Pt[s] &\Leftrightarrow \overline{s}(t) \in P^{\mathfrak{A}} \\ &\Leftrightarrow h(\overline{s}(t)) \in P^{\mathfrak{B}} \\ &\Leftrightarrow \overline{h \circ s}(t) \in P^{\mathfrak{B}} \\ &\Leftrightarrow \models_{\mathfrak{B}} Pt[h \circ s]. \end{aligned}$$

Other quantifier-free formulae without equality symbols can be proved by induction.

Proof of Homomorphism Theorem II

3. If h is one-to-one, we have

$$\begin{aligned} \models_{\mathfrak{A}} u \approx t[s] &\Leftrightarrow \overline{s}(u) = \overline{s}(t) \\ &\Leftrightarrow h(\overline{s}(u)) = h(\overline{s}(t)) \\ &\Leftrightarrow \overline{h \circ s}(u) = \overline{h \circ s}(t) \\ &\Leftrightarrow \models_{\mathfrak{B}} u \approx t[h \circ s]. \end{aligned}$$

Other cases are proved by induction.

4. By induction hypothesis, $\models_{\mathfrak{A}} \phi[s] \Leftrightarrow \models_{\mathfrak{B}} \phi[h \circ s]$ for any s.

$$\begin{aligned} \models_{\mathfrak{B}} \forall x \phi[h \circ s] &\Leftrightarrow \models_{\mathfrak{B}} \phi[(h \circ s)(x|b)] \text{ for every } b \in |\mathfrak{B}| \\ &\Leftrightarrow \models_{\mathfrak{B}} \phi[(h \circ s)(x|h(a))] \text{ for every } a \in |\mathfrak{A}| \\ &\Leftrightarrow \models_{\mathfrak{B}} \phi[h \circ (s(x|a))] \text{ for every } a \in |\mathfrak{A}| \\ &\Leftrightarrow \models_{\mathfrak{A}} \phi[s(x|a)] \text{ for every } a \in |\mathfrak{A}| \\ &\Leftrightarrow \models_{\mathfrak{A}} \forall x \phi[s]. \end{aligned}$$

Elementary Equivalence

- Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent ($\mathfrak{A} \equiv \mathfrak{B}$) if for every sentence σ,

$$\models_{\mathfrak{A}} \sigma \quad \Leftrightarrow \quad \models_{\mathfrak{B}} \sigma.$$

- By Homomorphism Theorem, two isomorphic structures are elementarily equivalent
  - but two elementarily equivalent structures are not necessarily isomorphic, e.g. $(\mathbb{R}, <_{\mathbb{R}})$ and $(\mathbb{Q}, <_{\mathbb{Q}})$
- The identity map from $(\mathbb{Z}^+, <^+_{\mathbb{Z}})$ into $(\mathbb{N}, <_{\mathbb{N}})$ is an isomorphism. We have

$$\models_{(\mathbb{Z}^+, <^+_{\mathbb{Z}})} \forall v_2(v_1 \not\approx v_2 \to v_1 < v_2)[\![v_1 \mapsto 1]\!]$$

but

$$\not\models_{(\mathbb{N}, <_{\mathbb{N}})} \forall v_2(v_1 \not\approx v_2 \to v_1 < v_2)[\![v_1 \mapsto 1]\!]$$

Generalization and Substitution

- A wff ϕ is a generalization of ψ if for some $n \geq 0$ and variables $x_1, \ldots, x_n$, $\phi = \forall x_1 \cdots \forall x_n \psi$
- For variable x and term t, write α^x_t for the formula obtained by replacing x with t. Formally,
  1. for atomic α, α^x_t is obtained by α by replacing the variable x by t;
  2. $(\neg\alpha)^x_t = (\neg\alpha^x_t)$;
  3. $(\alpha \to \beta)^x_t = (\alpha^x_t \to \beta^x_t)$;
  4. $(\forall y \alpha)^x_t = \begin{cases} \forall y \alpha & \text{if } x = y \\ \forall y(\alpha^x_t) & \text{if } x \neq y \end{cases}$
- t is substitutable for x in α if
  1. for atomic α, t is always substitutable for x in α;
  2. t is substitutable for x in $(\neg\alpha)$ if it is substitutable for x in α; t is substitutable for x in $(\alpha \to \beta)$ if it is substitutable for x in both α and β;
  3. t is substitutable for x in $\forall y \alpha$ if
     1. x does not occur free in $\forall y \alpha$; or
     2. y does not occur in t and t is substitable for x in α

More about Substitution

- Consider $\gamma = \forall v_2 B v_1 v_2$
- Then $\gamma^{v_1}_{v_2} = \forall v_2 B v_2 v_2$
  - however, v_2 is not substitutable for v_1 in γ (why?)
- When an axiom of the form $\forall x \alpha$ is instantiated, we have α^x_t for some term t
- But the substitution cannot be performed arbitrarily
  - thus we have to check whether t is substitutable for x in α

Logical Axioms Λ

The logical axioms Λ are generalizations of wffs of the following forms:

1. tautologies;
2. $\forall x \alpha \to \alpha^x_t$ where t is substitutable for x in α;
3. $\forall x(\alpha \to \beta) \to (\forall x \alpha \to \forall x \beta)$;
4. $\alpha \to \forall x \alpha$ where x does not occur free in α;
5. $x \approx x$;
6. $x \approx y \to (\alpha \to \alpha')$ where α is atomic and α' is obtained from α by replacing x in zero or more places by y

Modus Ponens

- (Modus ponens) From α and $\alpha \to \beta$, we may infer β:

$$\frac{\alpha,\ \alpha \to \beta}{\beta}$$

- A set Δ of formulae is closed under modus ponens if whenever α and $\alpha \to \beta$ are in Δ, then β is in Δ
- ϕ is a theorem of Γ ($\Gamma \vdash \phi$) if ϕ belongs to the set generated from $\Gamma \cup \Lambda$ by modus ponens

Definition

A deduction of ϕ from Γ is a sequence $\langle \alpha_0, \ldots, \alpha_n \rangle$ of formulae such that $\alpha_n = \phi$ and for each $i \leq n$,

- $\alpha_i \in \Gamma \cup \Lambda$; or
- for some $j, k < i$, α_i is obtained by modus ponens from α_j and $\alpha_k (= \alpha_j \to \alpha_i)$

Theorem and Deduction

Theorem

There exists a deduction of α from Γ iff α is a theorem of Γ.

Proof.

If there is a deduction $\langle a_0, \ldots, a_n \rangle$, then each α_i belongs to the set generated from $\Gamma \cup \Lambda$ by modus ponens. Hence $\Gamma \vdash \alpha_n (= \phi)$.
Conversely, every formula in $\Gamma \cup \Lambda$ has a deduction. Moreover, every formula obtained from $\Gamma \cup \Lambda$ by modus ponens has a deduction. Hence, every formula generated from $\Gamma \cup \Lambda$ by modus ponens has a deduction. Particularly, the theorem ϕ of Γ has a deduction. □

We therefore say ϕ is deducible from Γ if $\Gamma \vdash \phi$.

Tautologies

- A tautology in first-order logic is a wff obtained from a tautology in sentential logic by replacing each sentence symbol with a wff of first-order language

$$\forall x[(\forall y \neg Py \to \neg Px) \to (Px \to \neg \forall y \neg Py)]$$

is obtained from

$$(A \to \neg B) \to (B \to \neg A)$$

by generalization

More about Tautologies

- Divide wffs in first-order language in two groups
  1. Prime formulae are the atomic formulae and those of the form $\forall x \alpha$
  2. Non-prime formulae are those of the form $\neg \alpha$ or $\alpha \to \beta$
- Now take prime formulae as sentence symbols. Any tautology of the (new) sentential logic is a tautology in first-order language
- Consider $(\forall y \neg Py \to \neg Px) \to (Px \to \neg \forall \neg Px)$
  - there are two prime formulae: $\forall y \neg Py$ and Px
  - it remains to check whether $(A \to \neg B) \to (B \to \neg A)$ is a tautology
- By taking prime formulae as sentence symbols, first-order formulae are also wffs of sentential logic. Concepts for sentential logic are applicable.
  - it makes sense, for instance, to say "tautologically implies" in first-order language.

Deduction and Tautologically Implication

Theorem

$\Gamma \vdash \phi$ iff $\Gamma \cup \Lambda$ tautologically implies ϕ

Proof.

Observe that $\{\alpha, \alpha \to \beta\}$ tautologically implies β. Now suppose there is a truth assignment ν satisfying $\Gamma \cup \Lambda$. We can prove ν satisfies any theorem of Γ by induction on the length of deduction. The inductive step uses the observation.
Conversely, assume $\Gamma \cup \Lambda$ tautologically implies ϕ. By compactness theorem (for sentential logic), there is a finite subset $\{\gamma_1, \ldots, \gamma_m, \lambda_1, \ldots, \lambda_n\}$ tautologically implying ϕ. Hence,

$$\gamma_1 \to \cdots \to \gamma_m \to \lambda_1 \to \cdots \to \lambda_n \to \phi$$

is a tautology (why?) and hence in Λ. Applying modus ponens, we have ϕ. □

Examples of Theorems

- $\vdash Px \to \exists y Py$

$$\frac{\forall y \neg Py \to \neg Px \qquad (\forall y \neg Py \to \neg Px) \to (Px \to \neg \forall y \neg Py)}{Px \to \neg \forall y \neg Py}$$

- $\vdash \forall x(Px \to \exists y Py)$

$$\cfrac{\forall x(\forall y \neg Py \to \neg Px) \qquad \cfrac{\begin{array}{c}\forall x[(\forall y \neg Py \to \neg Px) \to \\ (Px \to \neg \forall y \neg Py)]\end{array} \qquad \begin{array}{c}\forall x(\alpha \to \beta) \to \\ (\forall x \alpha \to \forall x \beta)\end{array}}{\forall x(\forall y \neg Py \to \neg Px) \to \forall x(Px \to \neg \forall y \neg Py)}}{\forall x(Px \to \neg \forall y \neg Py)}$$

where α is $\forall y \neg Py \to \neg Px$ and β is $Px \to \neg \forall y \neg Py$.

Theorems and Metatheorems

- Note that the word “theorem” has two different meanings
- In $\Gamma \vdash \alpha$, we say α is a “theorem”
  - properties derived from Γ, at the object level
- We also say the following is a “theorem”

Theorem

$\Gamma \vdash \phi$ iff $\Gamma \cup \Lambda$ tautologically implies ϕ

  - properties about arbitrary Γ, at the meta level

Generalization Theorem I

Theorem (generalization)

If $\Gamma \vdash \phi$ and x does not occur free in any formula in Γ, then $\Gamma \vdash \forall x \phi$

Generalization Theorem II

Proof.

Fix a set Γ and a variable x not free in Γ. If $T = \{\phi : \Gamma \vdash \forall x \phi\}$ includes $\Gamma \cup \Lambda$ and is closed under modus ponens, then every theorem ϕ of Γ belongs to T. Hence $\Gamma \vdash \forall x \phi$ for any theorem ϕ.

- $\psi \in \Lambda$. Hence $\forall x \psi \in \Lambda$. Thus $\Gamma \vdash \forall x \psi$ and $\psi \in T$
- $\psi \in \Gamma$. Then x does not occur free in ψ. $\psi \to \forall x \psi \in \Lambda$ (axiom group 4). We have

$$\frac{\psi \qquad \psi \to \forall x \psi}{\forall x \psi}$$

- Suppose ϕ and $\phi \to \psi$. By induction hypothesis, $\Gamma \vdash \forall x \phi$ and $\Gamma \vdash \forall x(\phi \to \psi)$. We have

$$\cfrac{\forall x \phi \qquad \cfrac{\forall x(\phi \to \psi) \qquad \forall x(\phi \to \psi) \to (\forall x \phi \to \forall x \psi)}{\forall x \phi \to \forall x \psi}}{\forall x \psi}$$

□

Remark

- Informally, when we prove ______x______ from Γ and Γ does not restrict x, we should have $\forall x$______x______
  - this is exactly Generalization Theorem
- Axiom group 3 and 4 are crucial in the proof
- x must not occur free in Γ
  - $Px \not\models \forall x Px$, one should not have $Px \vdash \forall x Px$
- For applications, let us show $\forall x \forall y \alpha \vdash \forall y \forall x \alpha$
  By axiom group 2 (twice) and $\forall x \forall y \alpha$, we have $\forall x \forall y \alpha \vdash \alpha$. By applying Generalization Theorem (twice), we have $\forall x \forall y \alpha \vdash \forall y \forall x \alpha$

Rule T

Lemma (Rule T)

If $\Gamma \vdash \alpha_1, \ldots, \Gamma \vdash \alpha_n$ and $\{\alpha_1, \ldots, \alpha_n\}$ tautologically implies β, then $\Gamma \vdash \beta$

Proof.

$\alpha_1 \to \cdots \to \alpha_n \to \beta$ is a tautology and hence a logical axiom. Apply modus ponens. □

Deduction Theorem I

Theorem

If $\Gamma \cup \{\gamma\} \vdash \phi$, $\Gamma \vdash \gamma \to \phi$

Proof.

(First proof)

$$\begin{array}{rll} \Gamma \cup \{\gamma\} \vdash \phi & \text{iff} & \Gamma \cup \{\gamma\} \cup \Lambda \text{ tautologically implies } \phi \\ & \text{iff} & \Gamma \cup \Lambda \text{ tautologically implies } \gamma \to \phi \\ & \text{iff} & \Gamma \vdash \gamma \to \phi \end{array}$$

□

Deduction Theorem II

Proof.

(Second proof)
We show that $\Gamma \vdash \gamma \to \phi$ when $\Gamma \cup \{\gamma\} \vdash \phi$.

- $\phi = \gamma$. Clearly, $\Gamma \vdash \gamma \to \phi$
- $\phi \in \Lambda \cup \Gamma$. We have $\Gamma \vdash \phi$. Moreover, $\phi \to (\gamma \to \phi)$ is a tautology. (why?) By modus ponens, $\Gamma \vdash \gamma \to \phi$
- ϕ is obtained from ψ and $\psi \to \phi$ by modus ponens. By inductive hypothesis, $\Gamma \vdash \gamma \to \psi$ and $\Gamma \vdash \gamma \to (\psi \to \phi)$. Moreover, $\{\gamma \to \psi, \gamma \to (\psi \to \phi)\}$ tautologically implies $\gamma \to \phi$. By rule T, $\Gamma \vdash \gamma \to \phi$

□

Contraposition

Corollary (contraposition)

$\Gamma \cup \{\phi\} \vdash \neg\psi$ *iff* $\Gamma \cup \{\psi\} \vdash \neg\phi$

Proof.

$$\begin{aligned}\Gamma \cup \{\phi\} \vdash \neg\psi &\Rightarrow \Gamma \vdash \phi \to \neg\psi \text{ (Deduction Theorem)}\\ &\Rightarrow \Gamma \vdash \psi \to \neg\phi\\ &\qquad (\phi \to \neg\psi \text{ tautologically implies } \psi \to \neg\phi\text{, Rule T})\\ &\Rightarrow \Gamma \cup \{\psi\} \vdash \neg\phi \text{ (modus ponens)}\end{aligned}$$

The converse is obtained by symmetry. □

Inconsistency

- A set Γ of formulae is inconsistent if both $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$ for some β
- In this case, $\Gamma \vdash \alpha$ for any formula α
  - $\beta \to \neg\beta \to \alpha$ is a tautology

Reductio ad Absurdum

Corollary (reductio ad absurdum)

If $\Gamma \cup \{\phi\}$ *is inconsistent,* $\Gamma \vdash \neg\phi$.

Proof.

By Deduction Theorem, $\Gamma \vdash \phi \to \beta$ and $\Gamma \vdash \phi \to \neg\beta$ for some β. Moreover, $\{\phi \to \beta, \phi \to \neg\beta\}$ tautologically implies $\neg\phi$. □

Example

Example

Show $\vdash \exists x \forall y \phi \to \forall y \exists x \phi$

Proof.

$$\begin{aligned} & \vdash \exists x \forall y \phi \to \forall y \exists x \phi\\ \text{if} \quad & \exists x \forall y \phi \vdash \forall y \exists x \phi \text{ (Deduction Theorem)}\\ \text{if} \quad & \exists x \forall y \phi \vdash \exists x \phi \text{ (Generalization Theorem)}\\ \text{if} \quad & \neg\forall x \neg\forall y \phi \vdash \neg\forall x \neg\phi \text{ (Definition)}\\ \text{if} \quad & \forall x \neg\phi \vdash \forall x \neg\forall y \phi \text{ (contraposition)}\\ \text{if} \quad & \forall x \neg\phi \vdash \neg\forall y \phi \text{ (Generalization Theorem)}\\ \text{if} \quad & \{\forall x \neg\phi, \forall y \phi\} \text{ is inconsistent (reductio ad absurdum)}\\ \text{if} \quad & \forall x \neg\phi \vdash \neg\phi \text{ and } \forall y \phi \vdash \phi \text{ (axiom group 2)}\end{aligned}$$

□

Deduction Strategy

Given $\Gamma \vdash \phi$, how to find a proof of it?

- $\phi = (\psi \to \theta)$. This is the same as $\Gamma \cup \{\phi\} \vdash \theta$ (Deduction Theorem)
- $\phi = \forall x \psi$. This is the same as $\Gamma \vdash \psi$ after variable renaming (Generalization Theorem)
- ϕ is a negation.
  - $\phi = \neg(\psi \to \theta)$. This is the same as $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\theta$ (rule T)
  - $\phi = \neg\neg\psi$. This is the same as $\Gamma \vdash \psi$ (rule T)
  - $\phi = \neg\forall x \psi$. It suffices to show $\Gamma \vdash \neg\psi^x_t$ for some t substitutable for x in ϕ (reductio ad absurdum).
    - ★ but it is not always possible, e.g. $\vdash \neg\forall x \neg(Px \to \forall y Py)$
    - ★ this is case, we may use contraposition and reductio ad absurdum

Examples I

Example

If x does not occur free in α, show $\vdash (\alpha \to \forall x \beta) \leftrightarrow \forall x(\alpha \to \beta)$

Proof.

It suffices to show $\vdash (\alpha \to \forall x\beta) \to \forall x(\alpha \to \beta)$ and $\vdash \forall x(\alpha \to \beta) \to (\alpha \to \forall x\beta)$ (rule T).

- $\vdash (\alpha \to \forall x\beta) \to \forall x(\alpha \to \beta)$. It suffices to show $\{\alpha \to \forall x\beta, \alpha\} \vdash \beta$ (Deduction and Generalization Theorems). But this follows by modus ponens and axiom group 2
- $\vdash \forall x(\alpha \to \beta) \to (\alpha \to \forall x\beta)$. By Deduction and Generalization Theorems, it suffices to show $\{\forall x(\alpha \to \beta), \alpha\} \vdash \beta$. But this follows by axiom group 2 and modus ponens.

□

Examples II

Example (Eq2)

Show $\vdash \forall x \forall y(x \approx y \to y \approx x)$

Proof.

1. $\vdash x \approx y \to x \approx x \to y \approx x$. Ax 6
2. $\vdash x \approx x$. Ax 5
3. $\vdash x \approx y \to y \approx x$. 1, 2, T
4. $\vdash \forall x \forall y(x \approx y \to y \approx x)$. 3, gen

□

Note that this is not a formal proof of $\forall x \forall y(x \approx y \to y \approx x)$. This is an informal proof which shows that a formal proof exists

Examples III

Example

Show $\vdash x \approx y \to \forall z Pxz \to \forall z Pyz$

Proof.

1. $\vdash x \approx y \to Pxz \to Pyz$. Ax 6
2. $\vdash \forall z Pxz \to Pxz$. Ax 2
3. $\vdash x \approx y \to \forall z Pxz \to Pyz$. 1, 2, T
4. $\{x \approx y, \forall z Pxz\} \vdash Pyz$. 3, MP
5. $\{x \approx y, \forall z Pxz\} \vdash \forall z Pyz$. 4, gen
6. $\vdash x \approx y \to \forall z Pxz \to \forall z Pyz$ 5, ded

□

Generalization on Constants

Theorem

Assume that $\Gamma \vdash \phi$ and c is a constant symbol not in Γ. Then there is a variable y (not in ϕ) such that $\Gamma \vdash \forall y \phi^c_y$. Moreover, there is a deduction of $\forall y \phi^c_y$ from Γ where c does not appear.

Proof.

Let $\langle \alpha_0, \ldots, \alpha_n \rangle$ be a deduction of ϕ from Γ. Let y be a variable not in any of α_i's. We claim $\langle (\alpha_0)^c_y, \ldots, (\alpha_n)^c_y \rangle$ is a deduction of ϕ^c_y.

- $\alpha_k \in \Gamma$. Then $(\alpha_k)^c_y = \alpha_k \in \Gamma$.
- α_k is a logical axiom. Then $(\alpha_k)^c_y$ is also a logical axiom.
- α_k is obtained from α_i and $\alpha_j = \alpha_i \to \alpha_k$. Then $(\alpha_k)^c_y$ is obtained by $(\alpha_i)^c_y$ and $(\alpha_j)^c_y = (\alpha_i)^c_y \to (\alpha_k)^c_y$.

Thus, $\Gamma \vdash \phi^c_y$. By Generalization Theorem, $\Gamma \vdash \forall y \phi^c_y$. Moreover, c does not appear in the deduction of $\forall y \phi^c_y$ from Γ. □

Applications I

Corollary

Assume $\Gamma \vdash \phi^x_c$ and c does not occur in Γ or ϕ. Then $\Gamma \vdash \forall x \phi$ and there is a deduction of $\forall x \phi$ where c does not occur.

Proof.

By the previous theorem, there is a deduction of $\forall y (\phi^x_c)^c_y$ without c. Since c does not occur in ϕ, $(\phi^x_c)^c_y = \phi^x_y$. Observe that $(\forall y \phi^x_y) \to (\phi^x_y)^y_x$ is an axiom (axiom group 2). Moreover, $(\phi^x_y)^y_x = \phi$ (by induction). Thus, $\forall y \phi^x_y \vdash \forall x \phi$ (Generalization Theorem). □

Corollary (rule EI)

Assume c does not occur in ϕ, ψ, or Γ. If $\Gamma \cup \{\phi^x_c\} \vdash \psi$, then $\Gamma \cup \{\exists x \phi\} \vdash \psi$. Moreover, there is a deduction of ψ from $\Gamma \cup \{\exists x \phi\}$ without c.

Applications II

Proof.

By contraposition, we have $\Gamma \cup \{\neg\psi\} \vdash \neg\phi^x_c$. By the previous corollary, $\Gamma \cup \{\neg\psi\} \vdash \forall x \neg\phi$. Applying contraposition again, we have $\Gamma \cup \{\exists x \phi\} \vdash \psi$. □

"EI" stands for "existential instantiation."

Example

Example

Show $\vdash \exists x \forall y \phi \to \forall y \exists x \phi$

Proof.

$$
\begin{array}{ll}
 & \vdash \exists x \forall y \phi \to \forall y \exists x \phi \\
\text{if} & \exists x \forall y \phi \vdash \forall y \exists x \phi \text{ (Deduction Theorem)} \\
\text{if} & \forall y \phi^x_c \vdash \forall y \exists x \phi \text{ (rule EI)} \\
\text{if} & \forall y \phi^x_c \vdash \exists x \phi \text{ (Generalization Theorem)} \\
\text{if} & \phi^x_c \vdash \exists x \phi \ (\forall y \phi^x_c \vdash \phi^x_c \text{ and rule T}) \\
\text{if} & \forall x \neg\phi \vdash \neg\phi^x_c \text{ (contraposition)} \\
\text{if} & \vdash \forall x \neg\phi \to \neg\phi^x_c \text{ and } \forall x \neg\phi \vdash \forall x \neg\phi \text{ (MP)}
\end{array}
$$

□

Alphabetic Variants

Theorem

Let ϕ be a formula, t a term, and x a variable. Then there is a formula ϕ' such that (1) $\phi \vdash \phi'$ and $\phi' \vdash \phi$; (2) t is substitutable for x in ϕ'.

Proof.

Fix x and t. Construct ϕ' as follows. If ϕ is atomic, $\phi' = \phi$; $(\neg\phi)' = \neg\phi'$; and $(\phi \to \psi)' = \phi' \to \psi'$. Finally, define $(\forall y \phi)' = \forall z (\phi')^y_z$ where z is a fresh variable not in ϕ', t, or x. Note that t is substitutable for x in $(\phi')^y_z$ for z is fresh.
By inductive hypothesis, $\phi \vdash \phi'$. Thus $\forall y \phi \vdash \forall y \phi'$ (why?). Moreover, $\forall y \phi' \vdash (\phi')^y_z$. Hence $\forall y \phi' \vdash \forall z (\phi')^y_z$ by generalization. $\forall y \phi \vdash \forall z (\phi')^y_z$.
Conversely, $\forall z (\phi')^y_z \vdash ((\phi')^y_z)^z_y$. Since $((\phi')^y_z)^z_y = \phi'$ and $\phi' \vdash \phi$ (inductive hypothesis), $\forall z (\phi')^y_z \vdash \phi$. Finally, $\forall z (\phi')^y_z \vdash \forall y \phi$. □

Equality

- Eq1 $\vdash \forall x x \approx x$
- Eq2 $\vdash \forall x \forall y (x \approx y \to y \approx x)$
- Eq3 $\vdash \forall x \forall y \forall z (x \approx y \to y \approx z \to x \approx z)$
- Eq4 $\vdash \forall x_1 \forall x_2 \forall y_1 \forall y_2 (x_1 \approx y_1 \to x_2 \approx y_x \to P x_1 x_2 \to P y_1 y_2)$. Similarly for n-place predicates
- Eq5 $\vdash \forall x_1 \forall x_2 \forall y_1 \forall y_2 (x_1 \approx y_1 \to x_2 \approx y_x \to f x_1 x_2 \approx f y_1 y_2)$. Similarly for n-place functions

Soundness and Completeness

- Soundness.
$$\Gamma \vdash \phi \Rightarrow \Gamma \vDash \phi$$
- Completeness.
$$\Gamma \vDash \phi \Rightarrow \Gamma \vdash \phi$$

Substitution Lemma I

Lemma (Substitution)

If t is substitutable for x in ϕ, then

$$\vDash_{\mathfrak{A}} \phi^x_t[s] \textit{ iff } \vDash_{\mathfrak{A}} \phi[s(x|\overline{s}(t))].$$

Proof.

By induction on ϕ.

- ϕ is atomic. Consider, for instance,

$$\begin{aligned} \vDash_{\mathfrak{A}} P u^x_t[s] \quad & \text{iff} \quad \overline{s}(u^x_t) \in P^{\mathfrak{A}} \\ & \text{iff} \quad \overline{s(x|\overline{s}(t))}(u) \in P^{\mathfrak{A}} \text{ (induction on term } u\text{)} \\ & \text{iff} \quad \vDash_{\mathfrak{A}} P u[s(x|\overline{s}(t))] \end{aligned}$$

Substitution Lemma II

Proof (cont'd).

- $\phi = \neg\psi$ or $\psi \rightarrow \theta$. Follow by induction hypothesis.
- $\phi = \forall y \psi$ and x does not occur free in ϕ. Since ϕ^x_t is ϕ, the result follows.
- $\phi = \forall y \psi$ and x does occur free in ϕ. Since t is substitutable for x in ϕ, y does not occur in t. Hence $\overline{s}(t) = \overline{s(y|d)}(t)$ for any $d \in |\mathfrak{A}|$. Thus

$$\begin{array}{lll} \models_{\mathfrak{A}} \phi^x_t[s] & \text{iff} & \text{for all } d, \models_{\mathfrak{A}} \psi^x_t[s(y|d)] \\ & \text{iff} & \text{for all } d, \models_{\mathfrak{A}} \psi[s(y|d)(x|\overline{s(y|d)}(t))] \text{ (I.H.)} \\ & \text{iff} & \text{for all } d, \models_{\mathfrak{A}} \psi[s(y|d)(x|\overline{s}(t))] \\ & & (y \text{ does not occur in } t) \\ & \text{iff} & \models_{\mathfrak{A}} \phi[s(x|\overline{s}(t))]. \end{array}$$

□

Soundness Theorem I

Lemma

Every logical axiom is valid.

Proof.

We examine each axiom group as follows.

- Let $\mathfrak{A}$ be a structure and $s : V \rightarrow |\mathfrak{A}|$. Define a truth assignment ν on prime formulae γ by
$$\nu(\gamma) = T \text{ iff } \models_{\mathfrak{A}} \gamma[s].$$
Then $\overline{\nu}(\alpha) = T$ iff $\models_{\mathfrak{A}} \alpha[s]$ for any formula α. Particularly, if $\varnothing$ tautologically implies α, then $\models \alpha$.

Soundness Theorem II

Proof (cont'd).

- Consider, for example, $\forall x Px \rightarrow Pt$. Assume $\models_{\mathfrak{A}} \forall x Px[s]$. We have $\models_{\mathfrak{A}} Px[s(x|d)]$ for any $d \in |\mathfrak{A}|$. Particularly, $\models_{\mathfrak{A}} Px[s(x|\overline{s}(t))]$. By Substitution Lemma, $\models_{\mathfrak{A}} Pt[s]$. Thus $\models_{\mathfrak{A}} \forall x Px \rightarrow Pt$.
- Assume $\models_{\mathfrak{A}} \forall x(\alpha \rightarrow \beta)$ and $\models_{\mathfrak{A}} \forall x \alpha$. For any $d \in |\mathfrak{A}|$, $\models_{\mathfrak{A}} \alpha \rightarrow \beta[s(x|d)]$ and $\models_{\mathfrak{A}} \alpha[s(x|d)]$. Hence $\models_{\mathfrak{A}} \beta[s(x|d)]$ as required.
- Assume x does not occur free in α and $\models_{\mathfrak{A}} \alpha[s]$. Then $\models_{\mathfrak{A}} \alpha[s(x|d)]$ as required.
- Trivial, for $\models_{\mathfrak{A}} x \approx x[s]$ iff $s(x) = s(x)$.

Soundness Theorem III

Proof (cont'd).

- Assume α is atomic and α' is obtained from α by replacing x at some places by y. Suppose $\models_{\mathfrak{A}} x \approx y[s]$ and $\models_{\mathfrak{A}} \alpha[s]$. We have $s(x) = s(y)$. Hence for any term t and t' obtained from t by replacing x at some places y, we have $\overline{s}(t) = \overline{s}(t')$ by induction on t. The result follows by case analysis on α.

□

Soundness Theorem

Theorem

If $\Gamma \vdash \phi$, $\Gamma \vDash \phi$.

Proof.

By induction on the deduction.

- ϕ is a logical axiom. Hence $\vDash \phi$. Thus $\Gamma \vDash \phi$.
- $\phi \in \Gamma$. Clearly, $\Gamma \vDash \phi$.
- ϕ is obtained from ϕ and $\psi \to \phi$. By inductive hypothesis, $\Gamma \vDash \psi$ and $\Gamma \vDash \psi \to \phi$. Since $\{\psi, \psi \to \phi\}$ tautologically implies ϕ, we have $\Gamma \vDash \phi$.

□

Applications

Corollary

If $\vdash \phi \leftrightarrow \psi$, ϕ and ψ are logically equivalent.

Proof.

$\vdash \phi \to \psi$ implies $\phi \vdash \psi$ (modus ponens). Thus $\phi \vDash \psi$ (soundness). By symmetry, $\psi \vDash \phi$. □

Corollary

If ϕ' is an alphabetic variant of ϕ, ϕ and ϕ' are logically equivalent.

Proof.

By the definition of alphabetic variant. □

Satisfiability and Consistency

- We say Γ is satisfiable if there is some $\mathfrak{A}$ and s such that $\mathfrak{A}$ satisfies every member of Γ with s

Corollary

If Γ is satisfiable, Γ is consistent.

Proof.

Suppose Γ is inconsistent. Thus $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$ for some ϕ. By soundness theorem, $\Gamma \vDash \phi$ and $\Gamma \vDash \neg\phi$. Since Γ is satisfiable, $\vDash_{\mathfrak{A}} \phi[s]$ and $\vDash_{\mathfrak{A}} \neg\phi[s]$. □

Completeness Theorem

Lemma

The following are equivalent:

- *If $\Gamma \vDash \phi$, $\Gamma \vdash \phi$*
- *Any consistent set of formulae is satisfiable*

Proof.

Suppose Γ is a consistent set of formulae but Γ is not satisfiable. Since Γ is not satisfiable, we have $\Gamma \vDash \phi$ for any ϕ vacuously. Thus, $\Gamma \vdash \phi$ for any ϕ. Particularly, $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$. A contradiction.

Conversely, suppose $\Gamma \vDash \phi$. Then $\Gamma \cup \{\neg\phi\}$ is unsatisfiable and hence inconsistent. Thus $\Gamma \cup \{\neg\phi\} \vdash \psi$ and $\Gamma \cup \{\neg\phi\} \vdash \neg\psi$ for some ψ. We have $\Gamma \cup \{\neg\phi\} \vdash \psi \wedge \neg\psi$. By Deduction Theorem, $\Gamma \vdash \neg\phi \to (\psi \wedge \neg\psi)$. Note that $\vdash (\neg\phi \to (\psi \wedge \neg\psi)) \to \phi$ (why?). We have $\Gamma \vdash \phi$ by modus ponens. □

Completeness Theorem I

Theorem (Gödel, 1930)

Any consistent set of formulae is satisfiable.

Sketch. (Step 1).

Let Γ be a consistent set of wffs in a countable language.
Expand the language with a countably infinite set of new constant symbols. Then Γ remains consistent in the new language.

Details. (Step 1).

Otherwise, there is a β such that $\Gamma \vdash \beta \wedge \neg\beta$ in the new language. Since the deduction uses only finitely many new constants, we replace these new constants by variables (generalization on constants) and obtain β'. Then we have $\Gamma \vdash \beta' \wedge \neg\beta'$ in the original language. A contradiction.

Completeness Theorem II

Sketch. (Step 2).

For each wff ϕ in the new language and each variable x, consider wffs of the form

$$\neg\forall x\phi \to \neg\phi^x_c$$

where c is a new constant. We can have consistent $\Gamma \cup \Theta$ for some set Θ of wffs in such form.

Completeness Theorem III

Details. (Step 2).

Let $\langle\phi_1, x_1\rangle, \ldots, \langle\phi_n, x_n\rangle, \ldots$ be an enumeration. Define θ_n to be

$$\neg\forall x_n\phi_n \to \neg(\phi_n)^{x_n}_{c_n}$$

where c_n is the first new constant symbol not occurring in ϕ_n nor in θ_k for $k < n$. Let $\Theta = \{\theta_1, \ldots, \theta_n, \ldots\}$.
If $\Gamma \cup \Theta$ is inconsistent, there is a least $m \geq 0$ such that $\Gamma \cup \{\theta_1, \ldots, \theta_m, \theta_{m+1}\}$ is inconsistent (because deduction is finite). By RAA, $\Gamma \cup \{\theta_1, \ldots, \theta_m\} \vdash \neg\theta_{m+1}$. Let $\theta_{m+1} = \neg\forall x\psi \to \neg\psi^x_c$. Then

$$\Gamma \cup \{\theta_1, \ldots, \theta_m\} \vdash \neg\forall x\psi \quad \text{and} \quad \Gamma \cup \{\theta_1, \ldots, \theta_m\} \vdash \psi^x_c$$

Since c does not occur in $\Gamma \cup \{\theta_1, \ldots, \theta_m\}$, we have $\Gamma \cup \{\theta_1, \ldots, \theta_m\} \vdash \forall x\psi$ by generalization on constants. A contradiction to the minimality of m (or consistency of Γ).

Completeness Theorem IV

Sketch (Step 3).

We extend $\Gamma \cup \Theta$ to a maximal consistent set Δ such that for any wff ϕ either $\phi \in \Delta$ or $\neg\phi \in \Delta$. Observe that $\Delta \vdash \phi$ implies $\Delta \nvdash \neg\phi$ (consistency). Hence $\neg\phi \notin \Delta$. Thus $\phi \in \Delta$ (maximality).

Details (Step 3).

Let Λ be the set of logical axioms in the new language. Since $\Gamma \cup \Theta$ is consistent, there is no β such that $\Gamma \cup \Theta \cup \Lambda$ tautologically implies both β and $\neg\beta$ (why?). There is a truth assignment ν for prime formulae which satisfies $\Gamma \cup \Theta \cup \Lambda$ (why?). Define $\Delta = \{\phi : \overline{\nu}(\phi) = \mathrm{T}\}$. Then for any ϕ, either $\phi \in \Delta$ or $\neg\phi \in \Delta$. Moreover

$$\begin{aligned} \Delta \vdash \phi \quad &\Rightarrow \quad \Delta \cup \Lambda (= \Delta) \text{ tautologically implies } \phi \\ &\Rightarrow \quad \overline{\nu}(\phi) = \mathrm{T} \quad \Rightarrow \quad \phi \in \Delta. \end{aligned}$$

Δ cannot be inconsistent.

Completeness Theorem V

Sketch (Step 4).

Define a structure $\mathfrak{U}$ as follows

- $|\mathfrak{U}|$ = the set of all terms in the new language
- $\langle u, t\rangle \in E^{\mathfrak{U}}$ iff $u \approx t \in \Delta$
- For each n-place predicate symbol P, $\langle t_1, \ldots, t_n\rangle \in P^{\mathfrak{U}}$ iff $Pt_1 \cdots t_n \in \Delta$
- For each n-place function symbol f, define $f^{\mathfrak{U}}(t_1, \ldots, t_n) = ft_1 \cdots t_n$

Let $s : V \to |\mathfrak{U}|$ be the identity function. Then $\bar{s}(t) = t$ for all t. For any wff ϕ, let ϕ^* be the result of replacing all $\approx$ in ϕ by E. We have $\models_{\mathfrak{U}} \phi^*[s]$ iff $\phi \in \Delta$.

Completeness Theorem VI

Details (Step 4).

We prove $\models_{\mathfrak{U}} \phi^*[s]$ iff $\phi \in \Delta$ by induction. Difficult cases are:

- $\models_{\mathfrak{U}} Pt[s]$ iff $\bar{s}(t) \in P^{\mathfrak{U}}$ iff $t \in P^{\mathfrak{U}}$ iff $Pt \in \Delta$
- $\models_{\mathfrak{U}} (\neg\phi)^*[s]$ iff $\not\models_{\mathfrak{U}} \phi^*[s]$ iff $\phi \notin \Delta$ (I.H.) iff $\neg\phi \in \Delta$ (maximality)
- $$\begin{array}{rll} \models_{\mathfrak{U}} (\phi \to \psi)^*[s] & \text{iff} & \not\models_{\mathfrak{U}} \phi^*[s] \text{ or } \models_{\mathfrak{U}} \psi^*[s] \\ & \text{iff} & \phi \notin \Delta \text{ or } \psi \in \Delta \text{ (I.H.)} \\ & \text{iff} & \neg\phi \in \Delta \text{ or } \psi \in \Delta \\ & \Rightarrow & \Delta \vdash \phi \to \psi \text{ (rule T)} \\ & \Rightarrow & \phi \notin \Delta \text{ or } [\phi \in \Delta \text{ and } \Delta \vdash \psi] \text{ (case analysis)} \\ & \Rightarrow & \phi \notin \Delta \text{ or } \psi \in \Delta \end{array}$$

Completeness Theorem VII

Details (Step 4)(cont'd).

- Recall $\theta = \neg\forall x\phi \to \neg\phi^x_c \in \Delta$.

$$\begin{array}{rll} \models_{\mathfrak{U}} \forall x\phi^*[s] & \Rightarrow & \models_{\mathfrak{U}} \phi^*[s(x|c)] \\ & \Rightarrow & \models_{\mathfrak{U}} (\phi^*)^x_c[s] \text{ (substitution lemma)} \\ & \Rightarrow & \models_{\mathfrak{U}} (\phi^x_c)^*[s] \quad\Rightarrow\quad \phi^x_c \in \Delta \quad\Rightarrow\quad \neg\phi^x_c \notin \Delta \\ & \Rightarrow & \neg\forall x\phi \notin \Delta \ (\theta \in \Delta) \quad\Rightarrow\quad \forall x\phi \in \Delta. \end{array}$$

$$\begin{array}{rll} \not\models_{\mathfrak{U}} \forall x\phi^*[s] & \Rightarrow & \not\models_{\mathfrak{U}} \phi^*[s(x|t)] \text{ for some } t \\ & \Rightarrow & \not\models_{\mathfrak{U}} \psi^*[s(x|t)] \text{ for some alphabetic variant } \psi \\ & \Rightarrow & \not\models_{\mathfrak{U}} (\psi^x_t)^*[s] \text{ (substitution lemma)} \\ & \Rightarrow & \psi^x_t \notin \Delta \quad\Rightarrow\quad \forall x\psi \notin \Delta \ (\forall x\psi \to \psi^x_t \in \Delta) \\ & \Rightarrow & \forall x\phi \notin \Delta. \end{array}$$

Completeness Theorem VIII

Sketch (Step 5).

If Γ contains equality, consider the quotient structure $\mathfrak{U}/E$:

- Define $[t] = \{s : \langle s, t\rangle \in E^{\mathfrak{U}}\}$. Observe that $E^{\mathfrak{U}}$ is a congruence relation:
  - $E^{\mathfrak{U}}$ is a equivalence relation on $|\mathfrak{U}|$
  - $\langle t_1, \ldots, t_n\rangle \in P^{\mathfrak{U}}$ and $\langle t_i, t'_i\rangle \in E^{\mathfrak{U}}$ for $1 \le i \le n$, then $\langle t'_1, \ldots, t'_n\rangle \in P^{\mathfrak{U}}$
  - $\langle t_i, t'_i\rangle \in E^{\mathfrak{U}}$ for $1 \le i \le n$, then $\langle f^{\mathfrak{U}}(t_1, \ldots, t_n), f^{\mathfrak{U}}(t'_1, \ldots, t'_n)\rangle \in E^{\mathfrak{U}}$
- $|\mathfrak{U}/E| = \{[t] : t \text{ a term }\}$
- $\langle [t_1], \ldots, [t_n]\rangle \in P^{\mathfrak{U}/E}$ iff $\langle t_1, \ldots, t_n\rangle \in P^{\mathfrak{U}}$
- $f^{\mathfrak{U}/E}([t_1], \ldots, [t_n]) = [f^{\mathfrak{U}}(t_1, \ldots, t_n)]$. Particularly, $c^{\mathfrak{U}/E} = [c^{\mathfrak{U}}]$

Let $h(t) = [t]$ be the natural map from $|\mathfrak{U}|$ to $|\mathfrak{U}/E|$. h is a homomorphism of $\mathfrak{U}$ onto $\mathfrak{U}/E$. For any ϕ,

$$\phi \in \Delta \quad\Leftrightarrow\quad \models_{\mathfrak{U}} \phi^*[s] \quad\Leftrightarrow\quad \models_{\mathfrak{U}/E} \phi^*[h \circ s] \quad\Leftrightarrow\quad \models_{\mathfrak{U}} \phi[h \circ s]$$

Completeness Theorem IX

Details (Step 5).

Recall $\langle t, t' \rangle \in E^{\mathfrak{U}}$ iff $t = t' \in \Delta$ iff $\Delta \vdash t = t'$. Hence $E^{\mathfrak{U}}$ is a congruence relation on $\mathfrak{U}$, and both $P^{\mathfrak{U}/E}$ and $f^{\mathfrak{U}/E}$ are well-defined.
Clearly, h is a homomorphism of $\mathfrak{U}$ onto $\mathfrak{U}/E$. Moreover, $\langle [t], [t'] \rangle \in E^{\mathfrak{U}/E}$ iff $\langle t, t' \rangle \in E^{\mathfrak{U}}$ iff $[t] = [t']$. Thus

$$\begin{aligned} \phi \in \Delta \quad &\Leftrightarrow \quad \models_{\mathfrak{U}} \phi^*[s] \text{ (Step 4)} \\ &\Leftrightarrow \quad \models_{\mathfrak{U}/E} \phi^*[h \circ s] \text{ (homomorphism theorem)} \\ &\Leftrightarrow \quad \models_{\mathfrak{U}} \phi[h \circ s] \text{ (above)} \end{aligned}$$

Completeness Theorem X

Details (Step 6).

Restrict $\mathfrak{U}/E$ to the original language. The restricted $\mathfrak{U}/E$ satisfies every member of Γ with $h \circ s$. Γ is satisfiable. □

- Remark. If the original language is uncountable, a modified proof still works. We only add sufficiently many new constant symbols

Compactness Theorem

Theorem (Compactness)

1. *If $\Gamma \models \phi$, then $\Gamma_0 \models \phi$ for some finite $\Gamma_0 \subseteq \Gamma$;*
2. *If every finite subset Γ_0 of Γ is satisfiable, Γ is satisfiable.*

Proof.

1. Observe $\Gamma \models \phi$ implies $\Gamma \vdash \phi$. Since deductions are finite, $\Gamma_0 \vdash \phi$ for some finite $\Gamma_0 \subseteq \Gamma$. Hence $\Gamma_0 \models \phi$ by soundness theorem.
2. Suppose every finite subset of Γ is satisfiable, every finite subset of Γ is consistent (soundness theorem). Since deductions are finite, Γ is consistent. By completeness theorem, Γ is satisfiable.

□

History

- Kurt Gödel's 1930 doctoral dissertation contains the completeness theorem for countable languages. Compactness theorem was a corollary.
- Anatolii Mal'cev showed the compactness theorem for uncountable languages in 1941.
- Our proof of completeness theorem is based on Leon Henkin's 1949 dissertation.

References

1. Enderton. A Mathematical Introduction to Logic.