Logic
Solutions to Homework for Lecture II

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These are possible solutions to the homework for the first lecture. Some questions can be answered in more than one way, so if your answer differs from mine that does not mean you are wrong. In fact, my solution might be wrong, in which case you should contact me as soon as possible.

1 Propositional Logic

1. Prove the following using truth tables or the definition of validity and equivalence:

   (a) $P \wedge Q \to R \Leftrightarrow P \to Q \to R$

P	Q	R	$P \wedge Q$	$Q \to R$	$P \wedge Q \to R$	$P \to Q \to R$
F	F	F	F	T	T	T
F	F	T	F	T	T	T
F	T	F	F	F	T	T
F	T	T	F	T	T	T
T	F	F	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

   (b) $\models \bot \to P$

P	$\bot$	$\bot \to P$
F	F	T
T	F	T

   (c) $P \wedge Q \Leftrightarrow \neg(P \to \neg Q)$

P	Q	$\neg Q$	$P \to \neg Q$	$\neg(P \to \neg Q)$	$P \wedge Q$
F	F	T	T	F	F
F	T	F	T	F	F
T	F	T	T	F	F
T	T	F	F	T	T

(d) $P \to Q \Leftrightarrow \neg P \vee Q$

P	Q	$\neg P$	$P \to Q$	$\neg P \vee Q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

2. Show Neutrality and Boundedness (slide 27) using only the Important Equivalences, Idempotency, and algebraic reasoning.

   Neutrality:

$$\begin{array}{ll} & P \vee \bot \\ \Leftrightarrow & \{\text{by Complement}\} \\ & P \vee (P \wedge \neg P) \\ \Leftrightarrow & \{\text{by Absorption}\} \\ & P \end{array}$$

$$\begin{array}{ll} & P \wedge \top \\ \Leftrightarrow & \{\text{by Complement}\} \\ & P \wedge (P \vee \neg P) \\ \Leftrightarrow & \{\text{by Absorption}\} \\ & P \end{array}$$

   Boundedness:

$$\begin{array}{ll} & P \vee \top \\ \Leftrightarrow & \{\text{by Complement}\} \\ & P \vee (P \vee \neg P) \\ \Leftrightarrow & \{\text{by Associativity}\} \\ & (P \vee P) \vee \neg P \\ \Leftrightarrow & \{\text{by Idempotency}\} \\ & P \vee \neg P \\ \Leftrightarrow & \{\text{by Complement}\} \\ & \top \end{array}$$

$$\begin{array}{ll} & P \wedge \bot \\ \Leftrightarrow & \{\text{by Complement}\} \\ & P \wedge (P \wedge \neg P) \\ \Leftrightarrow & \{\text{by Associativity}\} \\ & (P \wedge P) \wedge \neg P \\ \Leftrightarrow & \{\text{by Idempotency}\} \\ & P \wedge \neg P \\ \Leftrightarrow & \{\text{by Complement}\} \\ & \top \end{array}$$

3. The connective $\overline{\wedge}$ ("nand") is defined by

$$\varphi \,\overline{\wedge}\, \psi := \neg(\varphi \wedge \psi)$$

(a) Draw a truth table for $P \barwedge Q$.

P	Q	$P \barwedge Q$
F	F	T
F	T	T
T	F	T
T	T	F

(b) Find formulas $\varphi_1, \varphi_2, \varphi_3$ with $\barwedge$ as their only connective such that

i. $\varphi_1 \Leftrightarrow \neg P$
Take, for example, $\varphi_1 := P \barwedge P$.

ii. $\varphi_2 \Leftrightarrow P \wedge Q$
In φ_1, take $P \barwedge Q$ for P; thus $\varphi_2 := (P \barwedge Q) \barwedge (P \barwedge Q)$.

iii. $\varphi_3 \Leftrightarrow \bot$
In φ_2, take $\neg P$ (i.e., φ_1) for Q; thus $\varphi_3 := (P \barwedge (P \barwedge P)) \barwedge (P \barwedge (P \barwedge P))$.

You do not have to provide truth tables.

Thus, $\{\barwedge\}$ is a functionally complete set!

2 First Order Logic

1. Can you find a signature in which $\forall x.\forall y.r(x, y) \wedge s(x) \rightarrow (\exists y.r(y))$ is a formula?

   No; r is used with two different arities.

2. Evaluate the following substitutions, indicating where you need alpha equivalence:

   - $(y \approx x \vee x \approx x \wedge (\forall y.x < y))[y/x]$

$$\begin{aligned}
& (y \approx x \vee x \approx x \wedge (\forall y.x < y))[y/x] \\
\equiv \quad & (y \approx x)[y/x] \vee (x \approx x)[y/x] \wedge (\forall y.x < y)[y/x] \\
\equiv \quad & y \approx y \vee y \approx y \wedge (\forall y.x < y)[y/x] \\
\equiv_\alpha \quad & y \approx y \vee y \approx y \wedge (\forall z.x < z)[y/x] \\
\equiv \quad & y \approx y \vee y \approx y \wedge (\forall z.y < z)
\end{aligned}$$

   - $(\forall u.\forall v.p(u) \rightarrow q(x) \wedge (\exists z.p(x) \wedge (\forall x.q(x, z))))[x/z]$

$$\begin{aligned}
& (\forall u.\forall v.p(u) \rightarrow q(x) \wedge (\exists z.p(x) \wedge (\forall x.q(x, z))))[x/z] \\
\equiv \quad & \forall u.\forall v.p(u) \rightarrow q(x) \wedge (\exists z.p(x) \wedge (\forall x.q(x, z)))[x/z] \\
\equiv \quad & \forall u.\forall v.p(u) \rightarrow q(x) \wedge (\exists z.p(x) \wedge (\forall x.q(x, z)))
\end{aligned}$$

3. Prove that $(\forall x.\varphi) \Leftrightarrow \neg(\exists x.\neg\varphi)$ for any formula φ.

   Let $\mathcal{M}$ be a model, σ a variable assignment. We want to show that $[\![\forall x.\varphi]\!]_{\mathcal{M},\sigma} = \mathtt{T}$ iff $[\![\exists x.\neg\varphi]\!]_{\mathcal{M},\sigma} = \mathtt{T}$.

"$\Rightarrow$": Assume $[\![\forall x.\varphi]\!]_{\mathcal{M},\sigma} = \mathtt{T}$; thus $[\![\varphi]\!]_{\mathcal{M},\sigma[x:=d]} = \mathtt{T}$ for any element d from the domain of $\mathcal{M}$. This means that $[\![\neg\varphi]\!]_{\mathcal{M},\sigma[x:=d]} = \mathtt{F}$ for any such d, hence $[\![\exists x.\neg\varphi]\!]_{\mathcal{M},\sigma} = \mathtt{F}$, and $[\![\neg\exists x.\neg\varphi]\!]_{\mathcal{M},\sigma} = \mathtt{T}$.

"$\Leftarrow$": Assume $[\![\neg\exists x.\neg\varphi]\!]_{\mathcal{M},\sigma} = \mathtt{T}$; thus $[\![\exists x.\neg\varphi]\!]_{\mathcal{M},\sigma} = \mathtt{F}$. This means that $[\![\neg\varphi]\!]_{\mathcal{M},\sigma[x:=d]} = \mathtt{F}$ for all elements d from the domain of $\mathcal{M}$, hence $[\![\varphi]\!]_{\mathcal{M},\sigma[x:=d]} = \mathtt{T}$ for all such d, and $[\![\forall x.\varphi]\!]_{\mathcal{M},\sigma} = \mathtt{T}$.